Pop Quiz (ok not really)

What is a linear combination of two vectors x_1 x_2

$$x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad x_2 = \begin{pmatrix} -1 \\ 5 \\ -2 \end{pmatrix}$$

In groups: Get me 4 different lin combos!

It's a vector of the form

$\beta_1 x_1 + \beta_2 x_2$ for some numbers β_1, β_2.

Challenge: find β_1 β_2 so $\beta_1 x_1 + \beta_2 x_2 = 0$.

(Some will find $\beta_1 = \beta_2 = 0$).

Ok, another? No

$$\beta_1 - \beta_2 = 0 \Rightarrow \beta_1 = \beta_2$$
$$2\beta_1 + 5\beta_1 = 0 \Rightarrow \beta_1 = 0.$$

Now tie into solving systems of equations

Find β_1, β_2 with $\beta_1 x_1 + \beta_2 x_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

Is same as find β_1 β_2

$$\begin{aligned} \beta_1 - \beta_2 &= 4 \\ 2\beta_1 + 5\beta_2 &= 5 \\ 3\beta_1 - 2\beta_2 &= 6 \end{aligned}$$

This idea is critical: we saw last class that solving systems of equations can be rephrased in the language of linear combinations.

$\mathrm{span}(x_1) = \{\beta_1 x_1 : \beta_1 \in \mathbb{R}\}$

$\mathrm{span}(x_1, x_2) = \{\beta_1 x_1 + \beta_2 x_2 : \beta_1, \beta_2 \in \mathbb{R}\}$

Last class:

$x_1, x_2, \ldots, x_n.$

These are linearly dependent if there exist numbers

$\beta_1, \beta_2, \ldots, \beta_n$ not all 0 such that

$\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n = 0$

(all β_i's $= 0$ always works.) An interesting linear combo lands on 0.

They are linearly independent if whenever $\beta_1, \ldots, \beta_n$ satisfy $\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n = 0$ all β_i's $= 0$.

We did examples!

Linear dependence is a property of a collection of vectors.

Observations

- Given $x_1, \ldots, x_n$, if any $x_i = 0$ then the collection is linearly dependent

E.g. if $x_1 = 0$, $\beta_1 = 1$, $\beta_i = 0$ otherwise.

- If $x_1, \dots, x_n$ are linearly dependent and we add in another vector x_{n+1} then $x_1, \dots, x_n, x_{n+1}$ are <u>also</u> lin dep.

$$\underbrace{\beta_1 x_1 + \dots + \beta_n x_n}_{\text{not all } 0} + 0\, x_{n+1} = 0$$

- Two vectors x_1, x_2 are lin dep iff one is a mult of other.

$\beta_1 x_1 + \beta_2 x_2 = 0$ WLOG $\beta_1 \neq 0$.

$$x_1 = -\frac{\beta_2}{\beta_1} x_2$$ so it's a mult.

And if $x_1 = \alpha x_2$ then $x_1 - \alpha x_2 = 0$

$\beta_1 = 1$ $\beta_2 = -\alpha$.

More generally:

The vectors $x_1, x_2, \dots, x_n$ are linearly dependent iff one of them is a linear combo of others.

E.g. $\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n = 0.$

Suppose $\beta_1 \neq 0.$

Then $x_1 = \underbrace{-\frac{\beta_2}{\beta_1} x_2 - \cdots - \frac{\beta_n}{\beta_1} x_n}_{\text{lin combo of } x_2, \ldots, x_n.}$

And if $x_1 = \gamma_2 x_2 + \cdots + \gamma_n x_n$ then

$$x_1 - \gamma_1 x_2 - \cdots - \gamma_n x_n = 0$$

$\beta_1 = 1, \beta_2 = -\gamma_1$ etc and not all $0.$

Can do this with any index.

Key property. Suppose $x_1, \ldots, x_n$ are lin ind.

Given a vector y, if we can find numbers $\beta_1, \ldots, \beta_n$ with $\beta_1 x_1 + \cdots + \beta_n x_n = y$,

these are <u>the</u> <u>only</u> numbers that work.

$$\beta_1 x_1 + \cdots + \beta_n x_n = y$$
$$\hat{\beta}_1 x_1 + \cdots + \hat{\beta}_n x_n = y$$

subtract

$$(\beta_1 - \hat{\beta}_1) x_1 + \cdots + (\beta_n - \hat{\beta}_n) x_n = 0$$

$$\beta_1 - \hat{\beta}_1 = 0, \ldots, (\beta_n - \hat{\beta}_n) = 0.$$

$$\Rightarrow \beta_i = \hat{\beta}_i \quad \forall i.$$

Bases

A basis for $\mathbb{R}^n$ is a collection of n linearly independant vectors.

$\mathbb{R}^2$ $\quad \underset{x_1}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \quad \underset{x_2}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$$\beta_1 x_1 + \beta_2 x_2 = 0$$

$$\beta_1 + \beta_2 = 0$$

$$2\beta_2 = 0 \quad \Rightarrow \beta_2 = 0$$

$$\Rightarrow \beta_1 = 0.$$

$\mathbb{R}^3$ $\quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ last class.

Why do we care?

Key fact: $n+1$ vectors in $\mathbb{R}^n$ are always linearly dependant.