This is a class called linear algebra.

We have seen no algebra!

$$3x + 2y - z = 5$$

$$-x + y + z = 2$$

$$2x + 2y + 3z = 9$$

Can we solve this? Is there a solution? How many?

Let me reframe this system

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} x + \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} y + \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} z = \begin{pmatrix} 5 \\ 2 \\ 9 \end{pmatrix}$$

Solving the system of equations is

the same as:

Find a linear combination of the vectors

$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ that equals $\begin{pmatrix} 5 \\ 2 \\ 9 \end{pmatrix}$

Now let me speak more generically.

Given vectors x_1, x_2, x_3 and y,

Can we find $\beta_1, \beta_2, \beta_3$ such that

$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = y \quad ?$$

Key property:

The vectors x_1, x_2 and x_3 are linearly dependent

if there are numbers β_1, β_2 and β_3 with

$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0.$$

Otherwise they are linearly independent.

(I.e. the only way

$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = 0 \text{ is}$$

the obvious one: $\beta_1 = \beta_2 = \beta_3 = 0$).

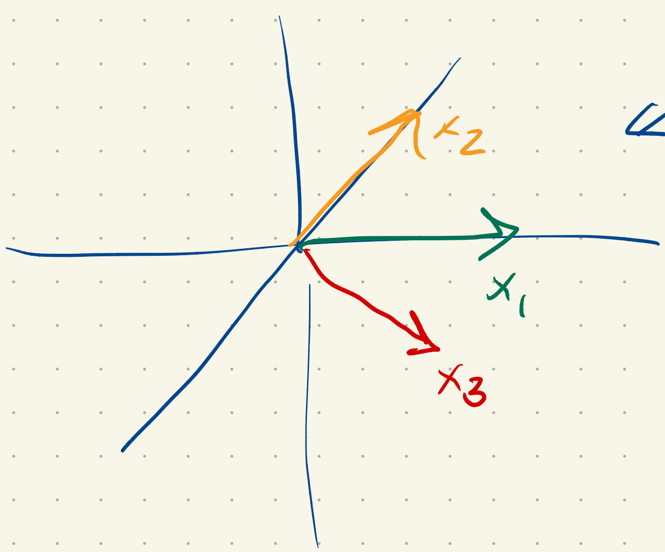
Linear independence is going to be a crucial property related to solving systems of equations.

E.g. $x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $x_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

These vectors are linearly dependent.

$$\beta_1 = -1, \quad \beta_2 = 1, \quad \beta_3 = 1$$

$$-x_1 + x_2 + x_3 = 0.$$



Graphically:

Intuitive: They are linearly dependent if one is a linear combination of the others

In this case $x_1 = x_2 + x_3$ or $x_3 = x_1 - x_2$.

Suppose we try to solve

$$\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\beta_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} \beta_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \beta_3 \\ -\beta_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\beta_1 + \beta_3 = 2$$

$$\beta_2 - \beta_3 = 1 \qquad \text{uh oh!}$$

$$0 = 3$$

<u>No solution</u> (3 equations 3 unknowns, no solutions)

Ok, fine. Suppose the RHS is $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$$\beta_1 + \beta_3 = 2$$
$$\beta_2 - \beta_3 = 1$$

I can see a solution: $\beta_1 = 2, \beta_2 = 1, \beta_3 = 0$ works. Great! But something's still not right.

$$\beta_1 = 2 - c$$
$$\beta_2 = 1 + c$$
$$\beta_3 = c$$

is a solution <u>for all</u> choices of the number c.

Like $c = 7$

$$\beta_1 = -5$$
$$\beta_2 = 8$$
$$\beta_3 = 7$$

$$\beta_1 + \beta_3 = -5 + 7 = 2 \checkmark$$
$$\beta_2 - \beta_3 = 8 - 7 = 1 \checkmark$$

So there are infinitely many solutions.

Both of these shortcomings stem from x_1, x_2, x_3

being linearly dependent,

You want to think of linear dependence as <u>bad</u>,
a kind of short coming.

easy example:

e_1, e_2 and e_3 in $\mathbb{R}^3$ are linearly independent.

Why? Need to show that if

$$\beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 = 0$$

then $\beta_1 = \beta_2 = \beta_3 = 0$.

$$\beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

so if this adds up to 0, $\beta_1 = 0$
$\beta_2 = 0$
$\beta_3 = 0$.

We can just read it off!

$x_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $x_2 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ are linearly independent.

Suppose $\beta_1 x_1 + \beta_2 x_2 = 0$

$$2\beta_1 - \beta_2 = 0$$
$$3\beta_1 + 5\beta_2 = 0$$

So $\beta_2 = 2\beta_1$. So $3\beta_1 + 10\beta_1 = 0$

So $13\beta_1 = 0$

So $\beta_1 = 0$. So $\beta_2 = 2\beta_1 = 0$.

How do you show vectors $x_1, x_2, \dots, x_n$ are linearly independent? It's <u>hard</u> in general!

You show that if you want to solve

$$\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n = 0$$

then $\beta_1 = \beta_2 = \dots = \beta_n = 0$. Period.