Last class:

Cauchy-Schwartz inequality

$$|x^T \cdot y| \leq \|x\| \|y\|.$$

I want to back up just a bit

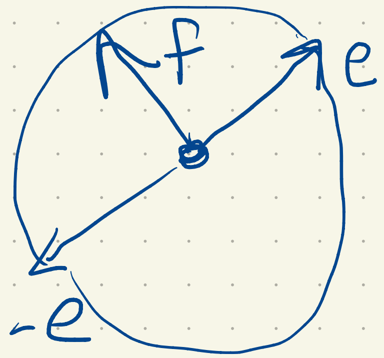
This is telling something about what the inner product measures. It tells us how alike x and y are?

How so? First suppose x and y are unit vectors, e, f. $(\|e\| = \|f\| = 1)$

$$|e^T f| \leq 1 \qquad -1 \leq e^T f \leq 1$$

Moreover: $e^T e = \|e\|^2 = 1$

$$e^T(-e) = -(e^T e) = -1$$



$e = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\quad e = (1, 0)$

$f = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\quad f = (0, 1)$

$e^T f = -\frac{1}{2} + \frac{1}{2} = 0$ $\quad e^T f = 0 + 0 = 0$

- If $e^T f = 0$ we say e and f are orthogonal (or perpendicular)

Morally, for unit vectors, $e^T f = 1$ means e and f are same

$e^T f = -1$ means e and f are opposite

$e^T f = 0$ means e and f are unrelated,

And Cauchy-Schwarz guarantees this kind of analysis holds in all dimensions.

For arbitrary vectors x, y, what does $x^T y$ tell you? $e = \frac{x}{\|x\|}$ $f = \frac{y}{\|y\|}$ are unit vectors ($x, y \neq 0$).

$$x = \|x\| e \, , \quad y = \|y\| f$$

$$x^T y = \underbrace{\underbrace{\|x\| \, \|y\|} \; \underbrace{\left(e^T f \right)}}$$

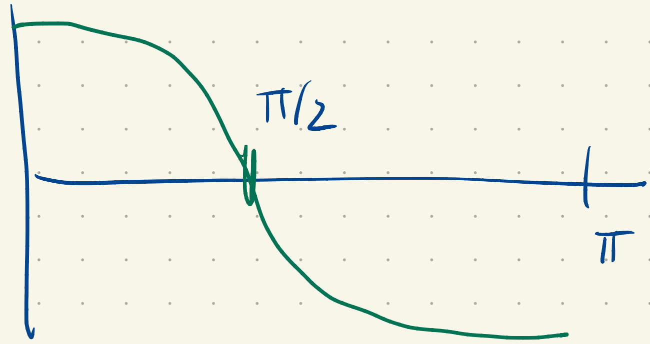
→ mixes these two pieces of information (magnitudes of x, y and their sameness.)

Oh, and if you are seeing $x^T y = \|x\| \|y\| \cos\theta$ you aren't wrong.

e, f are unit vectors we define

$$\angle(e, f) = \theta \qquad \theta = \arccos(e^T f)$$

$$\cos\theta = e^T f$$



$e^T f = 1 \Rightarrow \theta = 0$

$e^T f = -1 \Rightarrow \theta = \pi$

$e^T f = 0 = \theta =$

For arbitrary x, y we need to convert to unit vectors:

$$\cos\theta = \left(\frac{x}{\|x\|}\right)^T \left(\frac{y}{\|y\|}\right) = \frac{x^T y}{\|x\| \|y\|}$$

Moreover: $x^T y = \underbrace{\|x\| \|y\|}_{>0} \cos\theta$

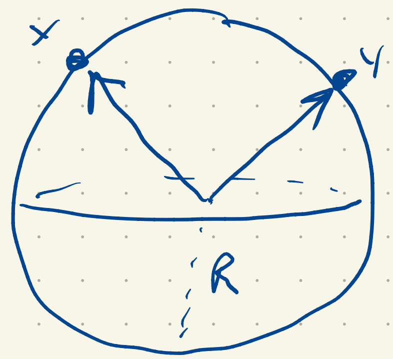
So the sign of $x^T y$ tells you about the sign of $\cos\theta$.

$x^T y > 0 \Rightarrow \theta$ is acute

$x^T y < 0 \Rightarrow \theta$ is obtuse

$x^T y = 0 \Rightarrow \theta = \frac{\pi}{2}$ (x, y are orthogonal)

How far, on the sphere, is it from x to y? $R\theta$.



circumference is $\underbrace{2\pi}_{\theta} R$ in rad!

$$R \arccos\left(\frac{x^T y}{R^2}\right) = d$$

I can't stress enough how fundamental the Cauchy-Schwartz inequality is.

The triangle inequality, $\|x+y\| \le \|x\| + \|y\|$ is a consequence,

$$\|x+y\|^2 = (x+y)^T(x+y) = \|x\|^2 + y^T x + x^T y + \|y\|^2$$

$$= \|x\|^2 + 2x^T y + \|y\|^2$$

$$\le \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

So $\|x+y\| \le \|x\| + \|y\|$ (oooh!)

Ok, can I show you that even in 147 dimensions the C-S inequality holds?

Lets do it for unit vectors e, f

$$0 \le \|e-f\|^2 = (e-f)^T(e-f)$$

$$= \|e\|^2 - 2f^Te + \|f\|^2$$

$$= 2 - 2e^Tf$$

So $\quad e^Tf \le 1.$

Since $-f$ is also a unit vector,

$$e^T(-f) \le 1 \Rightarrow -e^Tf \le 1 \Rightarrow e^Tf \ge -1.$$

So $\quad -1 \le e^Tf \le 1 \Rightarrow |e^Tf| \le 1.$

If $x, y \ne 0$ $\quad \left|\frac{x^T}{\|x\|}\frac{y}{\|y\|}\right| \le 1 \Rightarrow |x^Ty| \le \|x\|\,\|y\|.$

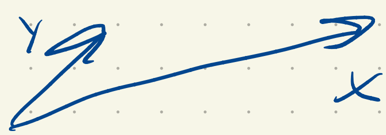
If $x = 0$ or $y = 0$, obvious.

Chapter 4 (it's a lab!)

Chapter 5 Linear Independence

(Now the real work begins).

Preview: Suppose I give you two vectors in $\mathbb{R}^3$



And I ask "what are all the vectors you can make by taking linear combinations of x and y?

What are all the vectors I can make forming

$\alpha x + \beta y$ for numbers α, β?

zero vector? yep! All multiples of x? yep!
of y? yep!

And indeed the whole plane that contains both x and y.

Now: what if I add $z = 3x - 2y$ into the mix?

What can I make forming linear combinations of x and y and z?

$$\alpha x + \beta y + \gamma z$$

Well I can always take $\gamma = 0$. So I get at least as much as before. Do I get anything new?

$$\alpha x + \beta y + \gamma(3x - 2y) = (\alpha + 3\gamma)x + (\beta - 2\gamma)y$$
$$= \alpha' x + \beta' y.$$

So no, nothing new.

The vectors x, y, and $3x - 2y$ are

called <u>linearly dependent</u>. It means they are

redundant from the point of view of making

linear combinations. I can throw one away and still make as many linear combos.

By contrast, $x = (1, 0, 0)$ and $y = (1, 1, 0)$

are called linearly dependent.

Throw one away and you can't form the x-y

plane.