A function $f: \mathbb{R}^n \to \mathbb{R}$

"from $\mathbb{R}^n$ to $\mathbb{R}$"

is <u>linear</u> if $f(x+y) = f(x) + f(y) \quad \forall x, y \in \mathbb{R}^n$

$f(\alpha x) = \alpha f(x)$ for $\alpha \in \mathbb{R}$
$x \in \mathbb{R}^n$

Non example $f(x,y) = x^2 - y^2$

$f(1,0) = 1$

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$f(2,0) = 4 \neq 2 = f(1,0) + f(1,0)$

Example $f(x,y) = 3x - 4y$

$$f(\underbrace{x_1, y_1}_{z_1}) + f(\underbrace{x_2, y_2}_{z_2}) = 3x_1 - 4y_1 + 3x_2 - 4y_2$$

$$= 3(x_1 + x_2) - 4(y_1 + y_2)$$

$$= f(z_1 + z_2)$$

For you: $f(\alpha z) = \alpha f(z)$

E.g. $a \in \mathbb{R}^n$, fixed

$$f(x) = a^T x \qquad (x \in \mathbb{R}^n)$$

$$f(x+y) = a^T(x+y)$$

$$= a_1(x_1+y_1) + \cdots + a_n(x_n+y_n)$$

$$= a_1x_1 + a_1y_1 + \cdots + a_n(x_n) + a_ny_n$$

$$= a^Tx + a^Ty$$

Similarly $f(\alpha x) = a^T \alpha x = \alpha a^T x = \alpha f(x)$,

So inner product against a fixed vector is linear.

Examples of linear functions:

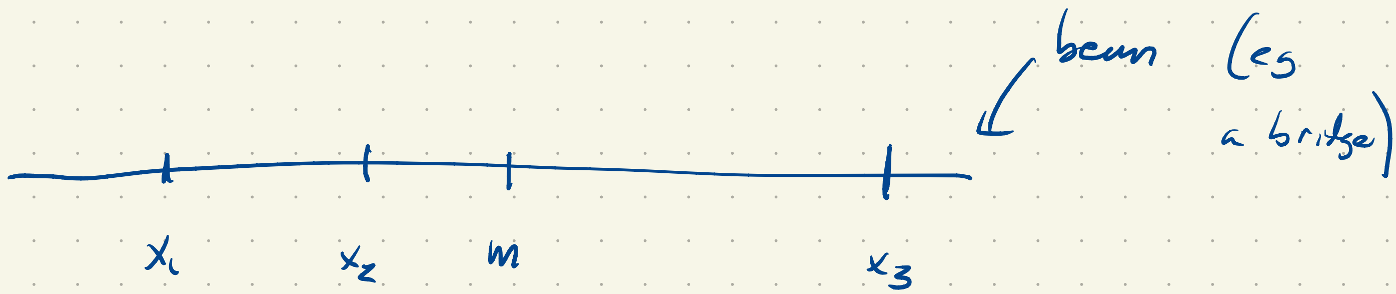
Given a time series, temps say

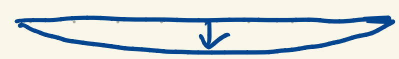
$$T = (T_1, \ldots, T_n)$$

tell me the temperature at time k.

$$f(T) = T_k$$

Text has a nice civil engineering example:



Three positions across the beam. 

Imagine point loads at x_1, x_2, x_3.

Want to measure the deflection (sag) s of the beam at the midpoint m as a consequence of weights w_1, w_2, w_3.

For a bridge: w_i in metric tons

s in mm

$$s(w_1, w_2, w_3) = c_1 w_1 + c_2 w_2 + c_3 w_3$$

$$= c^T w \qquad c = (c_1, c_2, c_3)$$

$$w = (w_1, w_2, w_3)$$

What are the units of c_i? mm / tonne

w_1	w_2	w_3	Measured sag	Predicted sag
1	0	0	0.12	—
0	1	0	0.31	—
0	0	1	0.26	—
0.5	1.1	0.3	0.481	0.479
1.5	0.8	1.2	0.736	0.740

Claim: every linear function $f: \mathbb{R}^n \to \mathbb{R}$ can be written

in the form $f(x) = c^T x$ for some $c \in \mathbb{R}^n$.

Why is that? Let $c_k = f(e_k)$ $\qquad e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$

slot k

$$x = (x_1, \ldots, x_n)$$

$$= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

$$f(x) = f(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$$

$$= f(x_1 e_1) + f(x_2 e_2) + \cdots + f(x_n e_n)$$

$$= x_1 f(e_1) + x_2 f(e_2) + \cdots + f(x_n e_n)$$

$$= c_1 x_1 + \cdots + c_n x_n$$

$$= c^T x$$

Suppose f is linear.

$$f(0) = f(0+0)$$

$$= f(0) + f(0)$$

$$\Rightarrow \boxed{f(0) = 0}$$

your favorite lines

$$f(x) = \underbrace{mx + b}_{\text{not linear unless } b = 0.}$$

If $f(x) = c_1 x_1 + \cdots + c_n x_n + b = c^T x + b$

we say f is <u>affine</u>.

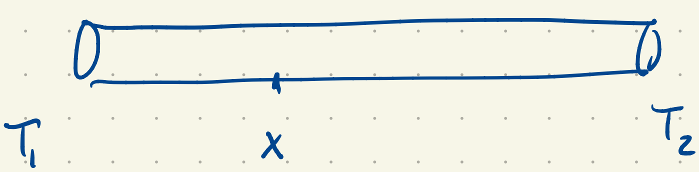
They satisfy a kind of limited superposition:

f is linear: $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

f is affine $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

if $\alpha + \beta = 1$

Sensitivities



non homogeneous

Insulated rod, fixed temps at each end.

I measure the temperature at the fixed position x (at steady state)

$f(T_1, T_2)$ is temp at x after waiting a while.

What happens if I change the temp at the left by a little bit ΔT_1? I expect a small change in temp at x, roughly proportional to the change ΔT_1

Ditto for T_2.

$$f(T_1 + \Delta T_1, T_2 + \Delta T_2) \approx f(T_1, T_2) + c_1 \Delta T_1 + c_2 \Delta T_2$$

The coefficients c_1, c_2 are called sensitivities.

$$L(\Delta T_1, \Delta T_2) = c_1 \Delta T_1 + c_2 \Delta T_2$$ is

a linear function $\quad c^T (\Delta T_1, \Delta T_2)$

in the perturbations ΔT_1 ΔT_2.

They act as scale factors.

In fact, you get these from calculus

$$f(x_1, x_2) = x_1 e^{x_2}$$

$$\nabla f = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} = \begin{bmatrix} e^{x_2} \\ x_1 e^{x_2} \end{bmatrix}$$

How you use it: Work at a point $x_0 = (3, 0)$

$$f(3,0) = 3$$

$$\nabla f = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\hat{f}(\underbrace{x_1, x_2}_{x}) = 3 + 1(x_1 - 3) + 3 \cdot (x_2 - 0)$$

$$= f(x_0) + \nabla f^T \underbrace{(x - x_0)}_{(\Delta x)}$$

$\hat{f}(x) \approx f(x)$ if x is close to x_0.

$\hat{f}(x_0) = f(x_0)$, perfect

$$\hat{f}(3.2, -0.1) = 3 + 1(0.2) + 3(-0.1)$$

$$= 3 + 0.2 - 0.3$$

$$= 2.9$$

(vs 2.895)

Only close to $(3,0)$, though.

$$f(1,-10) = 4.5 \times 10^{-5}$$

$$\hat{f}(1,-10) = 3 + (-2) + 3 \cdot (10-0)$$

$$= 31$$

$\hat{f}$ is an affine function. (though frequently called the linear approx)