Inner Product (duality)

$a = (a_1, a_2, a_3, a_4)$

$b = (b_1, b_2, b_3, b_4)$

$$a^T b = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

We'll see why T a bit later.

$a^T = [\ ,\ \dots,\ ]$

$a = \begin{bmatrix} \\ \\ \end{bmatrix}$

$(a^T)^T = a$

$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix}$

A <u>lot</u> of HW 1 is about seeing applications of this operation. What is it.

You can think of $a^T b$ as adding up the entries of b with weights coming from a,

e.g. $a = \vec{1}_4 \qquad b = (b_1, b_2, b_3, b_4)$

$$a^T b = b_1 + b_2 + b_3 + b_4$$

e.g $\quad a = e_3$

$$a^T b = b_3$$

e.g $\quad a = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$

$$a^T b = \frac{b_1 + b_2 + b_3 + b_4}{4} \quad \text{(average)}$$

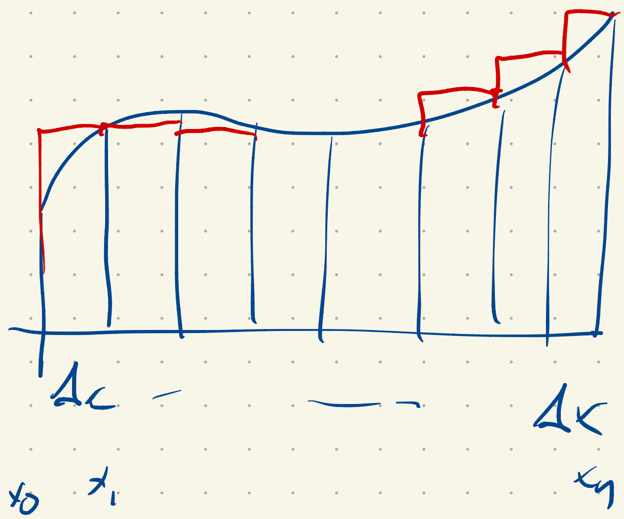
e.g. $\quad a$: portfolio assets

b: price per asset

$\underbrace{a_1 b_1}_{} + \cdots$] total value of portfolio

18 shares at \$46 a share
AAPL

e.g



$a = \Delta x \cdot \vec{1}$ $f(x_1)\Delta x + \cdots + f(x_n)\Delta x$ approx integral

$b_k = f(x_k)$

(total work, total energy prediction)

Some observations:

$$a^T b = b^T a$$

$$(\gamma a)^T b = \gamma (a^T b) \qquad a^T(\gamma b) = (\gamma b)^T a$$

$$= \gamma b^T a$$

$$(a+b)^T c = a^T c + b^T c \qquad = \gamma a^T b$$

$$a^T(b+c) = a^T b + a^T c$$

$$a^T a = a_1^2 + \cdots + a_n^2 \quad \text{sum of squares}$$

$x^2 + y^2 + z^2 \longrightarrow$ hmm. connection?

$a = (p_1, \ldots, p_n)$

probabilities $0 \le p_i \le 1$, $p_1 + \cdots + p_n = 1$

$b = (b_1, \ldots, b_n)$

outcomes, b_k with probability p_k

e.g. a draws, b_k is the prize value.

(one b_k is 0, and p_k close to 1)

$a^T b$ is the expected winnings.

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Last class: we introduced the duality operation (a.k.a. inner product)

$$a^T b = a_1 b_1 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k$$

And we saw that there are a number of natural operations that can be expressed like this.

Here's another:

$a = (a_1, \ldots, a_{n+1})$

t a number

$b = (1, t^1, t^2, \ldots, t^n)$

$a^T b = a_1 + a_2 t + \cdots + a_{n+1} t^n$ (polynomial evaluation)

$a^T b = b^T a$

$(a+b)^T c = a^T c + b^T c$

$a^T(\beta c) = \beta (a^T c)$

$(a+b)^T (c+d)$

$= a^T c + b^T c + a^T d + b^T d$

Recall your favorite integral rules:

$$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx$$

And your favorite derivative rules:

$$\frac{d}{dx}\Big(f(x) + g(x) \Big) = f'(x) + g'(x)$$

$$\frac{d}{dx} c f(x) = c f'(x)$$

~~It~~ you know the values of the pieces you can add them up and scale them.

Most mathematical operations don't work like this:

$$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A) \quad \text{e.g.}$$
$$= \sin(A) + \sin(B)$$

$$\sqrt{A+B} \neq \sqrt{A} + \sqrt{B}$$ (no matter what calc I students say ....)

But suppose $f(x) = 7x$

$$f(x+y) = 7(x+y)$$
$$= 7x + 7y$$

$$f(6x) = 7(6x)$$
$$= 7 \cdot 6 \cdot x$$
$$= 6 \cdot 7x$$
$$= 6\,f(x)$$

$$f(cx) = c\,f(x)$$

A function $f: \mathbb{R}^n \to \mathbb{R}$

"from $\mathbb{R}^n$ to $\mathbb{R}$"

is <u>linear</u> if

$$f(x+y) = f(x) + f(y) \quad \forall\, x, y \in \mathbb{R}^n$$

$$f(\alpha x) = \alpha f(x) \quad \text{for } \alpha \in \mathbb{R},\ x \in \mathbb{R}^n$$

Non example $\quad f(x,y) = x^2 - y^2$

$$f(1,0) = 1$$

$$f(1,0) = 1$$

$$f(2,0) = 4 \neq 2 = f(1,0) + f(1,0)$$

Example $\quad f(x,y) = 3x - 4y$

$$f(\underbrace{x_1, y_1}_{z_1}) + f(\underbrace{x_2, y_2}_{z_2}) = 3x_1 - 4y_1 + 3x_2 - 4y_2$$

$$= 3(x_1 + x_2) - 4(y_1 + y_2)$$

$$= f(z_1 + z_2)$$

For you: $f(\alpha z) = \alpha f(z)$

E.g. $a \in \mathbb{R}^n$, fixed

$$f(x) = a^T x \qquad (x \in \mathbb{R}^n)$$

$$\begin{aligned} f(x+y) &= a^T(x+y) \\ &= a_1(x_1+y_1) + \cdots + a_n(x_n+y_n) \\ &= a_1x_1 + a_1y_1 + \cdots + a_n(x_n) + a_ny_n \\ &= a^Tx + a^Ty \end{aligned}$$

Similarly $f(\alpha x) = a^T \alpha x = \alpha a^T x = \alpha f(x)$,

So inner product against a fixed vector is linear.

Examples of linear functions:

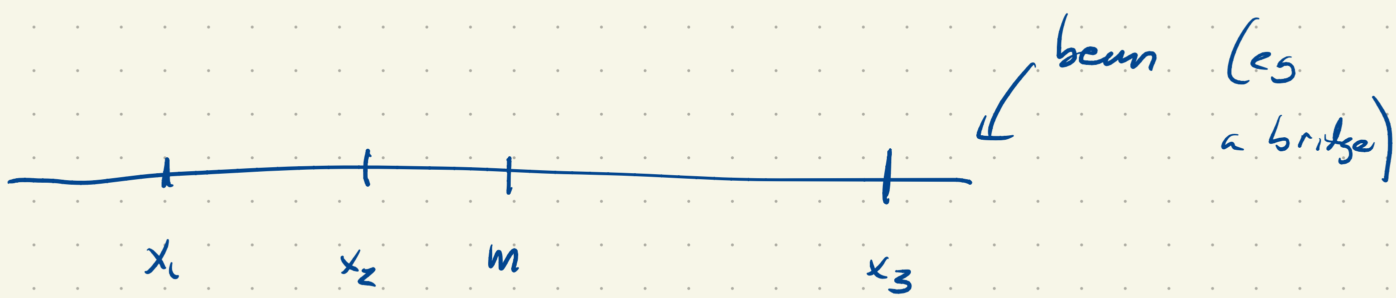
Given a time series, temps say

$$T = (T_1, \ldots, T_n)$$

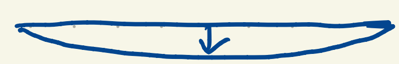
tell me the temperature at time k.

$$f(T) = T_k$$

Text has a nice civil engineering example:



Three positions across the beam.



Imagine point loads at x_1, x_2, x_3.

Want to measure the deflection (sag) s of the beam at the midpoint m as a consequence of weights w_1, w_2, w_3.

For a bridge: w_i in metric tons

s in mm

$$s(w_1, w_2, w_3) = c_1 w_1 + c_2 w_2 + c_3 w_3$$

$= c^T w$ $\quad c = (c_1, c_2, c_3)$

$w = (w_1, w_2, w_3)$

What are the units of c_i? mm / tonne

w_1	w_2	w_3	Measured sag	Predicted sag
1	0	0	0.12	—
0	1	0	0.31	—
0	0	1	0.26	—
0.5	1.1	0.3	0.481	0.479
1.5	0.8	1.2	0.736	0.740

Claim: every linear function $f : \mathbb{R}^n \to \mathbb{R}$ can be written

in the form $f(x) = c^T x$ for some $c \in \mathbb{R}^n$.

Why is that? Let $c_k = f(e_k)$ $\quad e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$

slot k

$$x = (x_1, \ldots, x_n)$$

$$= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

$$f(x) = f(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$$

$$= f(x_1 e_1) + f(x_2 e_2) + \cdots + f(x_n e_n)$$

$$= x_1 f(e_1) + x_2 f(e_2) + \cdots + f(x_n e_n)$$

$$= c_1 x_1 + \cdots + c_n x_n$$

$$= c^T x$$

Suppose f is linear.

$$f(0) = f(0+0)$$

$$= f(0) + f(0)$$

$$\Rightarrow \boxed{f(0) = 0}$$

Your favorite lines

$$f(x) = \underbrace{mx + b}_{\text{not linear unless } b = 0.}$$

If $f(x) = c_1 x_1 + \cdots + c_n x_n + b = c^T x + b$

we say f is <u>affine</u>.

They satisfy a kind of limited superposition:

f is linear: $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

f is affine: $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

if $\alpha + \beta = 1$