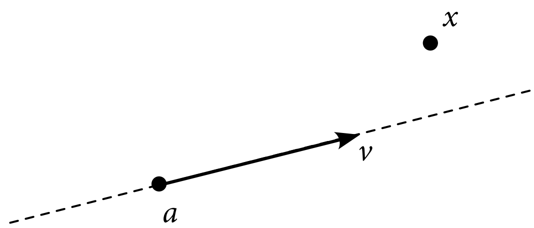
Name: Solutions

1. Let a be a vector and let v be a non-zero vector. As in the figure below, the line through a in the direction v is the set of all vectors of the form $a + sv$ where s is a number. Now consider some third vector x. We want to find the closest point z on the line to x.



a) The square of the distance from x to a point $a + sv$ on the line is given by

$$f(s) = ||a + sv - x||^2.$$

Carefully show that this square distance can **also** be written in the form

$$f(s) = ||a - x||^2 + 2s\, v^T(a - x) + s^2||v||^2$$

$$||a+sv-x||^2 = (a-x+sv)^T(a-x+sv)$$

$$= (a-x)^T(a-x) + s(a-x)^T v + sv^T(a-x) + s^2 v^T v$$

$$= ||a-x||^2 + 2s v^T(a-x) + s^2||v||^2$$

b) Recognizing this second expression for $f(s)$ as a quadratic in s, find the value of s that minimizes the square distance.

$$\frac{d}{ds}\, ||a-x||^2 + 2s v^T(a-x) + s^2||v||^2$$

$$= 2v^T(a-x) + 2s||v||^2$$

Set derivative $= 0$: $\quad 2v^T(a-x) + 2s||v||^2 = 0$

Solve for s $\quad s = \dfrac{-v^T(a-x)}{||v||^2}$

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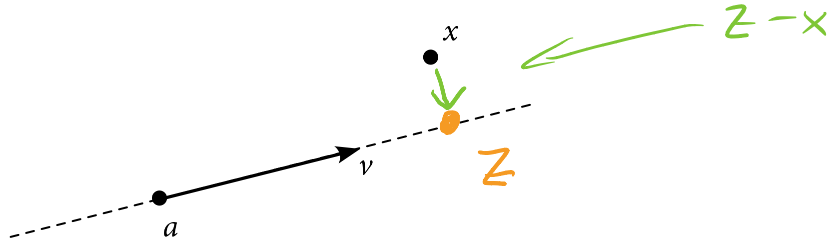

c) Using the value of s from part (b), one can show that the closest point z to x on the line is

$$z = a - \frac{1}{||v||^2}\left[(a-x)^T v\right] v.$$

You don't need to show this. However, use this expression for z to show that $z-x$ is perpendicular to v. Also, illustrate the perpendicularity by adding the point z in the figure above as well as the vector $z - x$.

$$z - x = (a-x) - \frac{1}{||v||^2}\left[(a-x)^T v\right] v$$

$$v^T(z-x) = v^T(a-x) - \frac{1}{||v||^2}\left((a-x)^T v\right) \underbrace{v^T v}_{||v||^2}$$

$$= v^T(a-x) - (a-x)^T v$$

$$= 0 \checkmark$$