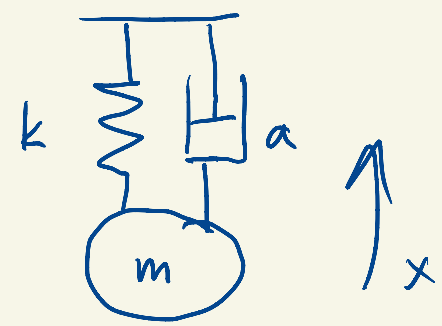
$$\dot{u} = a u$$

$$\frac{du}{dt} = a u \qquad u = c e^{at}$$

$$m, a, k \geq 0$$

$$m \ddot{x} + a \dot{x} + k x = 0$$

x position



t, time

$\dot{x}$ velocity $= V$

$$m\dot{v} + av + kx = 0$$

$$\dot{x} = v$$

$$\dot{x} = v$$

$$m\dot{v} = -av - kx$$

$$\dot{x} = 0 \cdot x + 1 \cdot v$$

$$\hat{v} = -\frac{k}{m}x - \frac{a}{m}v$$

$$w = \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\frac{d}{dt}w = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{a}{m} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\frac{dw}{dt} = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix}$$

$$\frac{d}{dt} w = A w \qquad A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix}$$

$$w = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{d}{dt} w = \underbrace{\begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix}}_{A} w$$

v

$$Av = \lambda v$$

$$w = f(t) v$$

$$\boxed{\frac{d}{dt} f = \lambda f}$$

$$f(t) = c e^{\lambda t}$$

$$\frac{d}{dt} w = \left(\frac{d}{dt} f\right) v$$

$$\begin{aligned} Aw &= A f(t) v \\ &= f(t) A v \\ &= \lambda f(t) v \end{aligned}$$

$$A = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} -2-\lambda & -1 \\ 2 & -5-\lambda \end{pmatrix}$$

$$= (\lambda+3)(\lambda+4)$$

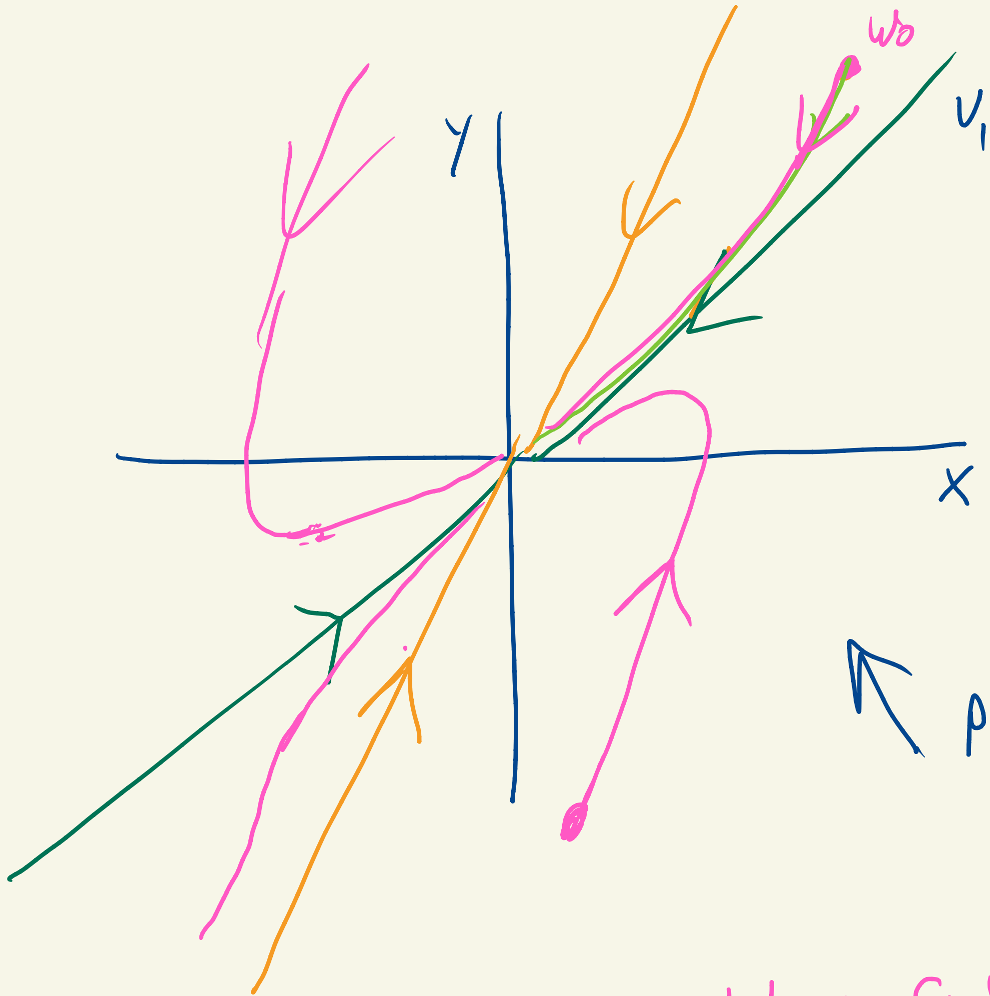
$$\lambda_1 = -3, \quad \lambda_2 = -4$$

$$\lambda_1 = -3, v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A - (-3)I = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda^n$$

$$A - (-4)I = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



phase portrait

$$w = ce^{-3t}v_1$$

$$w = ce^{-4t}v_2$$

$$w = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$w_0 = c_1 v_1 + c_2 v_2$$

$$w(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

$$= c_1 e^{-3t} v_1 + c_2 e^{-4t} v_2$$

$$= e^{-3t}\left(c_1 v_1 + c_2 e^{-t} v_2\right)$$

$$\frac{d}{dt} w = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} w$$

$$\underbrace{Av}_{} = \underbrace{\lambda v}_{}$$

$$\det \begin{bmatrix} 4-\lambda & -3 \\ 6 & -7-\lambda \end{bmatrix} = (\lambda+5)(\lambda-2)$$

$$\lambda_1 = -5$$

$$\lambda_2 = 2$$

$$v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

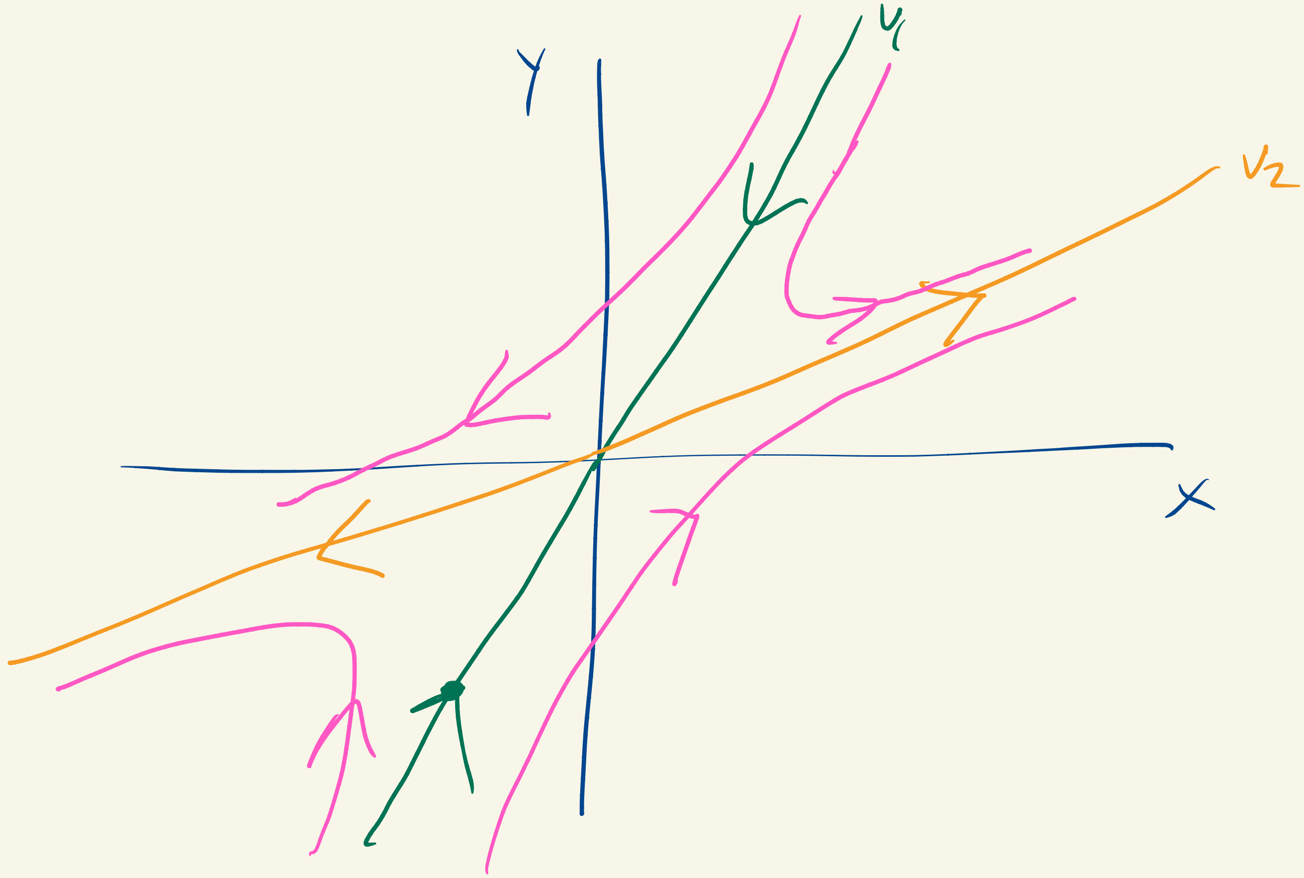
$\lambda_1 = -5 \qquad \lambda_2 = 2$

$$v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$w = ce^{-5t} v_1$

$w = ce^{2t} v_2$

$w = c_1 e^{-5t} v_1 + c_2 e^{2t} v_2$



$$m\ddot{x} + kx = 0$$

$$w = \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}}_{A}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 + \frac{k}{m}$$

$$\lambda = \pm\sqrt{\frac{k}{m}}\, i$$

eigenvectors: $\lambda = \sqrt{\frac{k}{m}}\, i$

$$A - \lambda I = \begin{bmatrix} -\sqrt{\frac{k}{m}}\, i & 1 \\ -\frac{k}{m} & -\sqrt{\frac{k}{m}}\, i \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ \sqrt{\frac{k}{m}}\, i \end{bmatrix} = \overset{a}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} - \overset{b}{\begin{bmatrix} 0 \\ -\sqrt{\frac{k}{m}} \end{bmatrix}} i$$

$$= a - ib$$

Complex solutions $\quad w = c e^{\lambda t} v$

Real and imaginary parts of w will be real solutions of the differential equation

$$\frac{d}{dt} \operatorname{Re}(w) = \operatorname{Re}\left(\frac{d}{dt} w\right)$$

$$\operatorname{Re}(Aw) = A \operatorname{Re}(w) \qquad (A \text{ is real})$$

$$\frac{d}{dt} w = Aw$$

$$\frac{d}{dt} \operatorname{Re} w = \operatorname{Re}\left(\frac{d}{dt} w\right) = \operatorname{Re}(Aw) = A \operatorname{Re}(w)$$

$$\lambda = \sqrt{\frac{k}{m}}\, i$$

$$v = \begin{bmatrix} 1 \\ \sqrt{\frac{k}{m}}\, i \end{bmatrix} = \overset{a}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} - \overset{b}{\begin{bmatrix} 0 \\ -\sqrt{\frac{k}{m}} \end{bmatrix}} i$$

$$w = e^{\lambda t}(a - ib)$$

$$= e^{i\sqrt{k/m}\,t}(a - ib) = \left(\cos\left(\sqrt{k/m}\,t\right) + i\sin\left(\sqrt{k/m}\,t\right)\right)(a - ib)$$

$$= \left(\cos\left(\sqrt{k/m}\,t\right)a + \sin\left(\sqrt{k/m}\,t\right)b\right)$$

$$- i\left(\cos\left(\sqrt{k/m}\,t\right)b - \sin\left(\sqrt{k/m}\,t\right)a\right)$$

$$w_1(t) = \cos(\omega t)\, a + \sin(\omega t)\, b$$

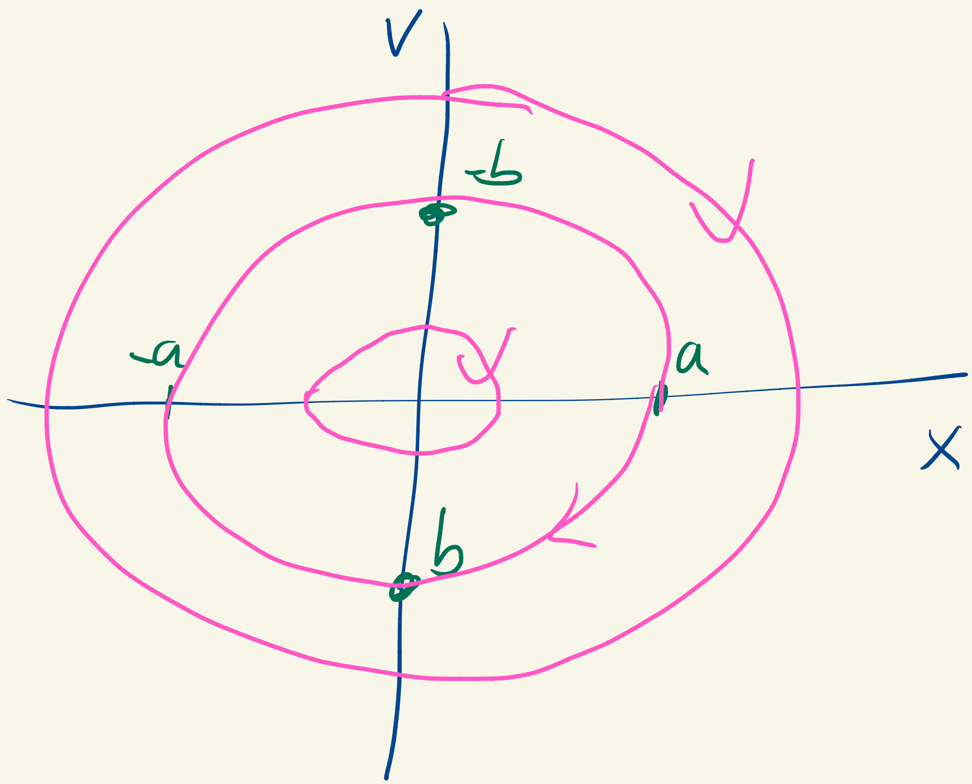
$$w_2(t) = \cos(\omega t)\, b - \sin(\omega t)\, a$$

$$w = c_1 w_1(t) + c_2 w_2(t)$$

$$\omega = \sqrt{k/m}$$

$$a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ -\sqrt{k/m} \end{bmatrix}$$



$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{a}{m} \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -k/m & -\frac{a}{m} - \lambda \end{pmatrix}$$

$$= \lambda\left(\frac{a}{m} + \lambda\right) + \frac{k}{m}$$

$$= \lambda^2 + \frac{a}{m}\lambda + \frac{k}{m}$$

$$= \frac{1}{m}\left[m\lambda^2 + a\lambda + k \right] \rightarrow \text{charac. poly}$$

$$m\ddot{x} + a\dot{x} + kx = 0$$

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4km}}{2m}$$

$$= -\frac{a}{2m} \pm \frac{1}{2m}\sqrt{a^2 - 4km}$$

$$-\alpha \pm \beta i$$

$$e^{-\alpha t} e^{\pm \beta i t} \quad \checkmark$$