$$\left[\begin{array}{c|c} a_{11} & * \cdots * \\ \hline w & \boxed{*} \end{array}\right] \leadsto \left[\begin{array}{c|c} a_{11} & * \cdots * \\ \hline 0 & \boxed{*'} \end{array}\right]$$

$$\left[\begin{array}{c|c} 0 & * \cdots * \\ \hline w & \boxed{*} \end{array}\right]$$

$w = 0$?

$w \neq 0$?

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \leadsto \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & 0 & \square & * & \cdots & * \\ 0 & \cdots & 0 & \square & & & \\ 0 & & \cdots & & & 0 & \square \\ 0 & & \cdots & & & & 0 \\ 0 & & \cdots & & & & 0 \end{bmatrix}$$

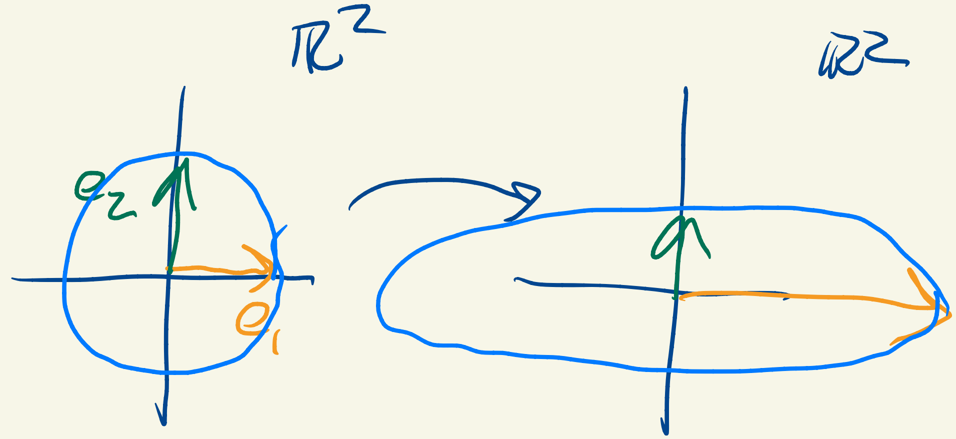
Eigenvalues:

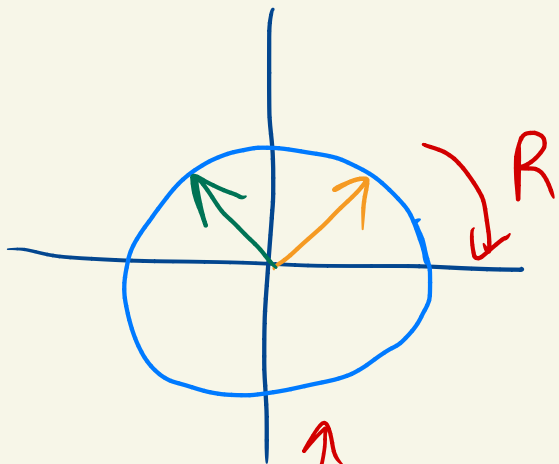
D

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

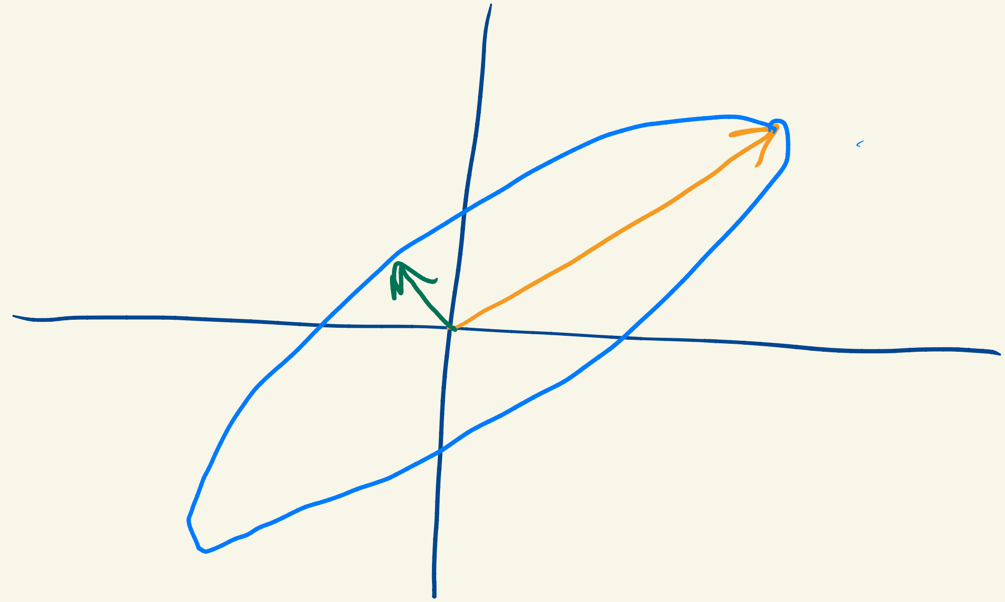
$$x \longmapsto Ax$$

$$\mathbb{R}^2 \qquad\qquad \mathbb{R}^2$$





$R^{-1}DR$



$$A\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{4}\begin{bmatrix}12\\12\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$$

$$A=\frac{1}{4}\begin{bmatrix}7&5\\5&7\end{bmatrix}$$

$$A\begin{bmatrix}-1\\1\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}-1/2\\1/2\end{bmatrix}$$

How to detect?

$Ax = 3x \qquad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Ax = \frac{1}{2}x \qquad x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

a number

$\underbrace{\text{eigen}}_{\text{"right" "correct"}}\text{value}$

$Ax = \lambda x$

eigenvector $\neq 0$

$Ax = \lambda I x$

$$\underbrace{(A - \lambda I)}\, x = 0$$

$Bx = 0 \quad x \neq 0$

square

such an x exists iff B is singular

$\neq 0$

(no inverse exists)

(cols are lin. dep.)

B is singular iff $\det B = 0$

$$\det(A - \lambda I) = 0$$

To find the eigenvalues we look for λ's where

$$\det(A - \lambda I) = 0.$$

$$A = \frac{1}{4}\begin{bmatrix} 7 & 5 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 7/4 & 5/4 \\ 5/4 & 7/4 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 7/4 - \lambda & 5/4 \\ 5/4 & 7/4 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \left(7/4 - \lambda\right)\left(\frac{7}{4} - \lambda\right) - \left(\frac{5}{4}\right)^2$$

"characteristic polynomial of A"

$$\left(\frac{7}{4} - \lambda\right)\left(\frac{7}{4} - \lambda\right) - \left(\frac{5}{4}\right)^2 = 0$$

quadratic in λ

$$\lambda - \frac{7}{4} = \pm \frac{5}{4}$$

$$\lambda = \frac{7}{4} \pm \frac{5}{4} = \underbrace{3, \tfrac{1}{2}}_{\text{eigenvalues!}}$$

$$\underbrace{(A - 3I)}_{\text{nullspace}} x = 0 \qquad Ax = 3Ix$$

$$Ax = 3x$$

$$\begin{bmatrix} \frac{7}{4}-3 & \frac{5}{4} \\ \frac{5}{4} & \frac{7}{4}-3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{4} & \frac{5}{4} \\ \frac{5}{4} & -\frac{5}{4} \end{bmatrix}$$

$$\begin{bmatrix} -5/4 & 5/4 \\ \cancel{4} & \cancel{4} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$-\tfrac{5}{4}a + \tfrac{5}{4}b = 0$$

$$-a + b = 0$$

$a=1$, $b=1$ works

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector.

$x = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$ — . — —— . .

Exercise: show $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ~~is~~ is an eigenvector for eigenvalue 1/2.

In general: $A: n \times n$

$$\det(A - \lambda I) = 0$$

polynomial of degree n $\Big]$ $p(\lambda)$

$p(\lambda) = 0$

$\pm \lambda^n + \dots = 0$

$(3-\lambda)^2 = 0$

$$p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

roots, eigenvalues.

For each λ_k, the associated eigenvectors are the null space of $A - \lambda_k I$

$$A = \begin{bmatrix} 2 & 2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Find eigenvalues, eigenvectors

$$Ae_3 = A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 0 \\ -2 & -3-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (5-\lambda)\det\begin{pmatrix} 2-\lambda & 2 \\ -2 & -3-\lambda \end{pmatrix}$$

$$= (5-\lambda)\big((2-\lambda)(-3-\lambda) + 4\big)$$

$$= (5-\lambda)(-6+\lambda+\lambda^2+4)$$

$$= (5-\lambda)(\lambda^2+\lambda-2)$$

$$= (5-\lambda)(\lambda+2)(\lambda-1)$$

eigenvalues: $5, -2, 1$

$$A+2I = \begin{bmatrix} 4 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \xrightarrow{R_2+\frac{1}{2}R_1} \begin{bmatrix} 4 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\rightarrow \begin{array}{c} \begin{array}{ccc} p & f & p \end{array} \\ \begin{bmatrix} 4 & 2 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

x_2 is free. $x_2 = 1$ and solve for x_1, x_3