$$Ax = b$$

1) $Lw = b$ forward subs

2) $Ux = w$ back subs

Last class we hinted at and your HW expands on:

$A \leftarrow$ square matrix.

For many of these we can factor A

$$A = L\,U$$

lower triangular

1's on diagonal

upper triangular

$$\begin{bmatrix} 3 & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \to \begin{bmatrix} 3 & * & * \\ 0 & 0 & * \\ 0 & * & * \end{bmatrix}$$

$$\begin{bmatrix} & * & * \\ 0 & & * \\ 0 & 0 & \end{bmatrix}$$

Sometimes we need to interchange rows.

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$XA = A$ with rows 1 and 2 swapped

$$A = \begin{pmatrix} r_1^T \\ r_2^T \\ r_3^T \end{pmatrix}$$

$$XA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1^T \\ r_2^T \\ r_3^T \end{pmatrix} = \begin{pmatrix} r_2^T \\ r_1^T \\ r_3^T \end{pmatrix}$$

exchange matrix

$X_1, \ldots, X_k$ exchange matrices

$$\underbrace{X_1 X_2 \cdots (X_k}_{P} A)$$

permutation matrices

rows of I

in some order

A : square matrix.

$$\underline{PA = LU}$$ where P is a permutation matrix

rows of P are orthonormal

$$PP^T = I \Rightarrow P^T = P^{-1} \Rightarrow P^T P = I$$

$\Rightarrow$ cols are o.n.

$$A = P^T L U$$

$$Ax = b$$

$$PAx = Pb$$

$$LUx = Pb$$

1) Solve $Lw = Pb$ (forwards subs)

2) $Ux = w$ (back subs.)

no zeros on diag. of U.

$$\begin{aligned} Ax &= P^T L U x \\ &= P^T L w \\ &= P^T P b \\ &= b \checkmark \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinants

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$ad-bc = \det(A)$$

If $\det(A) \neq 0$ then A has an inverse

If $\det(A) = 0$ then one can show (Exercise) that the columns of A are linearly dependent and hence A does not have an inverse.

determinants are defined for square matrices of all sizes

Three basic properties

1) $\det(I) = 1$ $\qquad \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1\cdot 1 - 0\cdot 0 = 1$ ✓

2) If X is an exchange matrix

$$\det(XA) = -\det(A)$$

$$\det\begin{pmatrix} c & d \\ a & b \end{pmatrix} = cb - ad = -(ad-bc) = -\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ ✓

3) det is linear in each row separately

$$\det\begin{pmatrix} \alpha a & \alpha b \\ c & d \end{pmatrix} = \alpha ad - \alpha bc = \alpha(ad-bc)$$
$$= \alpha \det(A)$$

$$\det\begin{pmatrix} a+a' & b+b' \\ c & d \end{pmatrix} = (a+a')d - (b+b')c$$
$$= ad + a'd - bc - b'c$$
$$= ad - bc + a'd - b'c$$
$$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \det\begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 1+6 & 2-4 \\ 3 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 7 & -2 \\ 3 & 4 \end{pmatrix}}_{A} \qquad \det(A) = 28 + 6 = 34$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 6 & -4 \\ 3 & 4 \end{pmatrix}$$

$$4 - 6 = -2 \qquad 24 + 12 = 36$$

$$-2 + 36 = 34$$

$$\boxed{(\vec{x} + \vec{y}) \times \vec{z} = \vec{x} \times \vec{z} + \vec{y} \times \vec{z}}$$

$$\det(v_1, v_2, \dots, v_n) = \det\left(\begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{pmatrix}\right)$$

$$\uparrow$$

$$\in \mathbb{R}^n$$

$$\det(v_2, v_1, v_3, \dots, v_n) = -\det(v_1, v_2, v_3, \dots, v_n)$$

$$\det(\alpha v_1, v_2, \dots, v_n) = \alpha \det(v_1, v_2, \dots, v_n)$$

$$\det(v_1 + v_1', v_2, \dots, v_n) = \det(v_1, v_2, \dots, v_n) + \det(v_1', v_2, \dots, v_n)$$

$$\det(e_1, e_2, \dots, e_n) = \underline{1}$$

Consequences

$$\det\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} = 0 \cdot d - 0 \cdot c = 0$$

a) If A has a zero row then

$$\det A = 0$$

$$\det(0, v_2, \dots, v_n) = \det(2 \cdot 0, v_2, \dots, v_n) = 2 \det(0, v_2, \dots, v_n)$$

$$\Rightarrow \det(0, v_2, \dots, v_n) = 0$$

b) If two rows of A are the same then

$$\det(A) = 0$$

$$\underbrace{\det(v_1, v_1, v_3, \dots, v_n)} = -\underbrace{\det(v_1, v_1, v_3, \dots, v_n)}$$

$$2\det(v_1, v_1, v_3, \dots, v_n) = 0$$

$$\det(v_1, 5v_1, v_3, \dots, v_n) = 5\det(v_1, v_1, v_3, \dots, v_n)$$

$$= 0$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

e) $\det(v_1, v_2 + \alpha v_1, v_3, \dots, v_n)$

$$= \det(v_1, v_2, v_3, \dots, v_n) + \det(v_1, \alpha v_1, v_3, \dots, v_n)$$

$$= \det(v_1, v_2, v_3, \dots, v_n) + \alpha \det(v_1, v_1, v_3, \dots, v_n)$$

$$= \underline{\hspace{6cm}} + 0$$

$$= \det(v_1, v_2, v_3, \dots, v_n)$$

f) If the rows of A are linearly dependant

then $\det A = 0$

Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \neq 0$ and

some $\alpha_i \neq 0$. $\qquad \alpha_1 \neq 0$

$$v_1 = \frac{1}{\alpha_1}\left[-\alpha_2 v_2 - \cdots - \alpha_n v_n\right]$$

$$\det(v_1, v_2, \ldots, v_n) = \det\left(\frac{1}{\alpha_1}\left[-\alpha_2 v_2 - \cdots - \alpha_n v_n\right], v_2, \ldots, v_n\right)$$

$$= \frac{1}{\alpha_1}\Big(\det(-\alpha_2 v_2, v_2, \ldots, v_n) + \det(-\alpha_3 v_3, v_2, v_3, \ldots, v_n)$$

$$+ \cdots + \det(-\alpha_n v_n, v_2, \ldots, v_n)\Big)$$

$$= \frac{1}{\alpha_1}\left(0 + 0 + \cdots + 0\right)$$

$$= 0$$

c) If X is an exchange matrix

$$\det(X) = -\det(I) = -\underline{1}$$

d) If P is a permutation matrix

$$\det(P) = \begin{cases} 1 \\ -1 \end{cases}$$