$$N(A) = \{x : Ax = 0\}$$

Why care? Suppose you solve $Ax = b$

Suppose $v \in N(A)$.

Then $x + v$ is another solution

$$A(x+v) = b$$

has to do with A

has to do with b

Moreover, if x_1 and x_2 both solve $Ax_i = b$

then $x_2 = x_1 + v$ where $v \in N(A)$.

$$A(x_2 - x_1) = 0$$

If A is wide and if the rows of A are linearly independent we can always solve

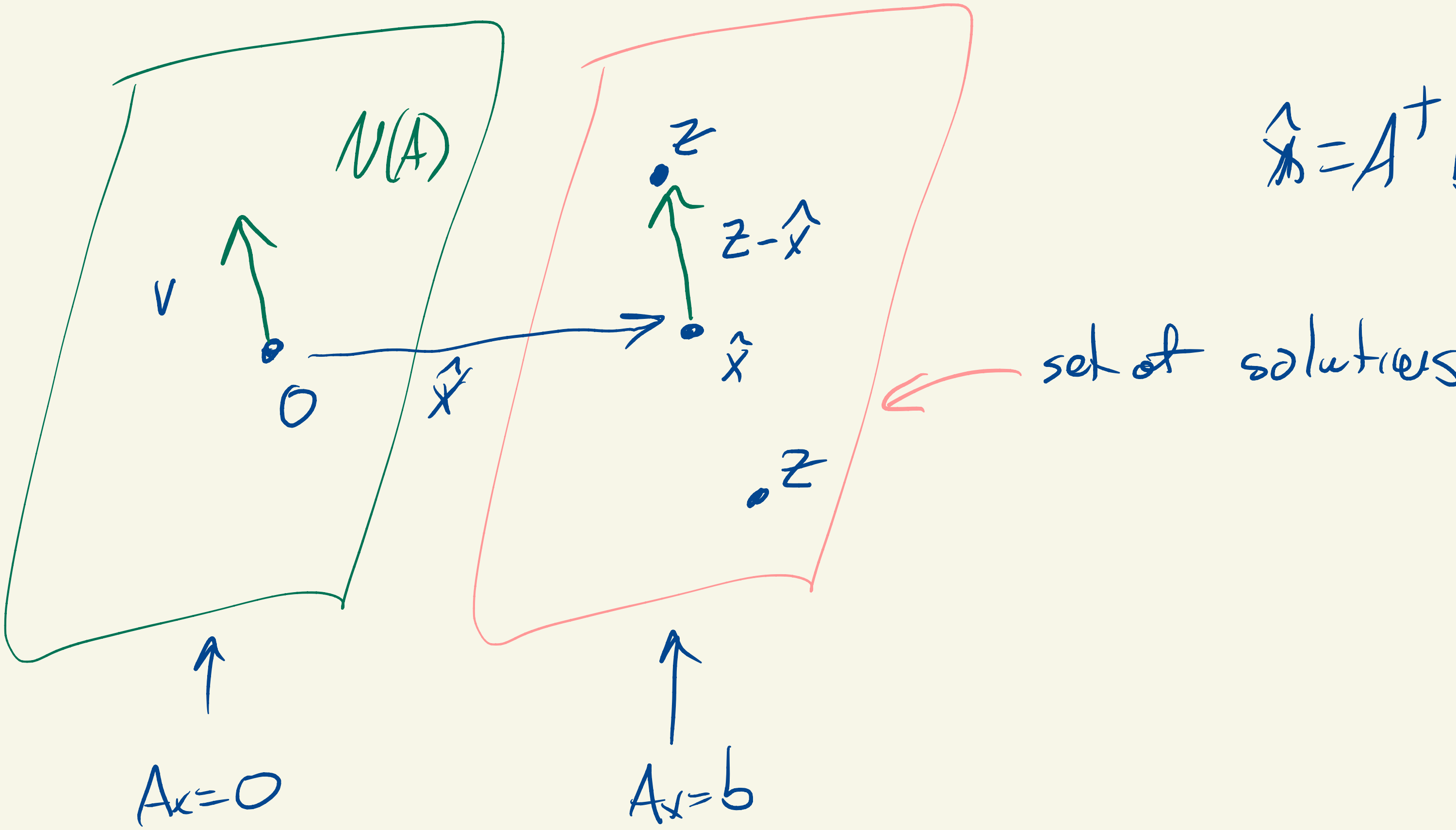
$$Ax = b$$

A has a right inverse, $A^+ = A^T(AA^T)^{-1}$

I claim $x = A^+b$ is a solution.

Indeed $\quad Ax = AA^+b = Ib = b$ ✓

But because A is wide the columns of A are not linearly independant so $N(A)$ is not trivial.



$$\hat{x}=A^{\dagger}b$$

Claim: If $\hat{x}=A^{\dagger}b$ and if z is another solution of $Az=b$ then $\|\hat{x}\| \leq \|z\|$

$$\|z\|^2 = \|\hat{x} + (z-\hat{x})\|^2 = \|\hat{x}\|^2 + \underbrace{2\hat{x}^T(z-\hat{x})}_{=0} + \|z-\hat{x}\|^2$$

$(a+b)^T(a+b)$

Recall $\quad \hat{x} = A^+ b = A^T \underbrace{(AA^T)^{-1}}_{C} b$

$$\hat{x} = A^T C b$$

So $\quad \hat{x}^T(z-\hat{x}) = (A^T C b)^T (z-\hat{x})$

$$= b^T C^T A (z-\hat{x})$$

$$= b^T C^T [b - b]$$

$$= 0$$

$$\|z\|^2 = \|\hat{x}\|^2 + \|z-\hat{x}\|^2$$

$$\|z\| \geq \|\hat{x}\|$$

How to find $N(A)$?

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Easy case is A is row echelon, a cousin of upper-triangular

$$\begin{bmatrix} p & x & x & x & x & x & x \\ 0 & p & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & p & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$p \neq 0$

pivot

p's are different

x are anything.

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$A \qquad\qquad x$

x_1, x_3 pivot variables.

x_2 free variable.

"special element of $N(A)$ associated with x_2"

I'll set $x_2 = 1$ and solve for x_1 and x_3

with $A\underset{\sim}{x} = 0$.

$$x_1 + 2x_2 + 3x_3 = 0$$

$$0x_1 + 0x_2 + 4x_3 = 0$$

$$0x_1 + 0x_2 + 0x_3 = 0$$

irrelevant.

$$x_1 + 3x_3 = -2$$

$$4x_3 = 0$$

$x_3 = 0$

$x_2 = 1$

$x_1 = -2$

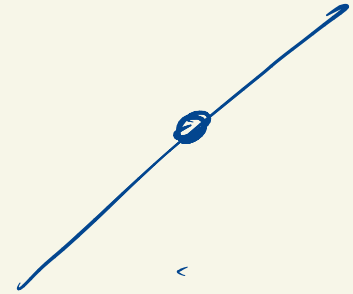
$$v = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$cv \in N(A)$

for all numbers

c.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



pivots

$$A = \begin{array}{c} \begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2$

$x_3 + 2x_4 + 3x_5 = 0 \Rightarrow x_3 = 0$

$x_5 = 0$

Two special solutions $\quad x_2 = 1 \quad x_2 = 0$

$x_4 = 0 \quad x_4 = 1$

$x_1 + 3 \cdot (-2) + 4 = 0$

$x_5 = 0$

$x_4 = 0$

$x_3 = 0$

$x_2 = 1$

$x_1 = -2$

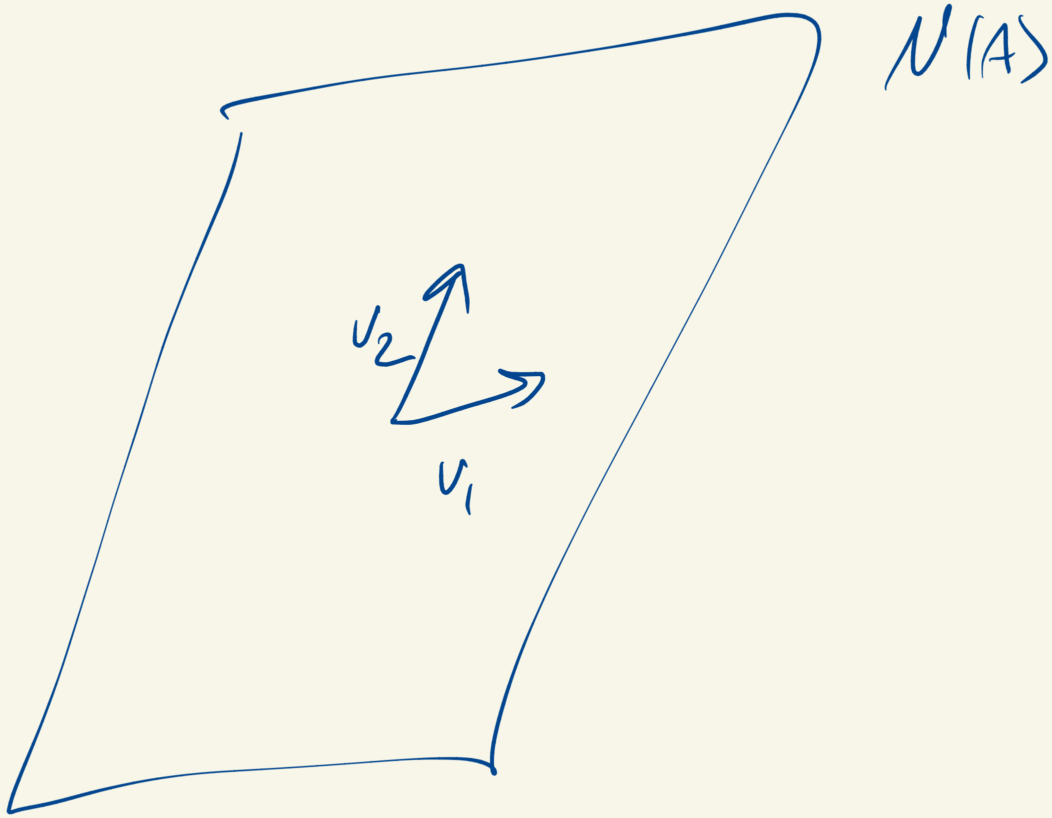
$$\begin{array}{cc} v_1 & v_2 \\ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \end{array}$$

$c_1 v_1 + c_2 v_2 = 0$

$$\begin{bmatrix} * \\ c_1 \\ * \\ c_2 \\ * \end{bmatrix} = 0$$

$$N(A) = \{ c_1 v_1 + c_2 v_2 : \; c_1, c_2 \in \mathbb{R} \}$$

$\hookrightarrow$ all liner combinations of v_1 and v_2.



$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$