The null space of a matrix

Recall that the columns of a matrix A are linearly independent if and only if the only solution of $Ax=0$ is $x=0$.

Well, what if there are interesting (nonzero) solutions of $Ax=0$?

This can only happen if the columns of A are not linearly independent (i.e. they are linearly dependent)

This is rare when A is tall or square

but always happens when A is wide.

Def: The <u>null space of</u> A is the set

$N(A)$ of all vectors x with $Ax = 0$.

(<u>kernel</u>)

$$\boxed{0 \in N(A)}$$

If $N(A) = \{0\}$

I'll say "the null space is trivial"

Why care?

$$\underbrace{\begin{bmatrix} 1 & 3 & 7 \\ 2 & 4 & 10 \end{bmatrix}}_{A} x = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

One solution: $x = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$

Consider $v = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $\quad Av = 0 \quad v \in N(A)$

Consider $x + v = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

$$A(x+v) = Ax + Av$$
$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 0$$

$$A(x+2v) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Principle 1) $cv \in N(A)$ for all numbers c.

$Av = 0 \quad A(cv) = cAv = c0 = 0.$

2) If $Ax = b$ and if $Av = 0$

then $A(x+v) = Ax + Av = b.$

Ideally: want to solve $Ax = b$ $N(A)$ is not trivial

1) Find one solution $\hat{x}$

2) Determine all the things in $N(A)$

Every solution is $\hat{x} + v$ with $v \in N(A)$.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \underbrace{N(A)}_{\mathbb{R}^2} = \mathbb{R}^2$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \qquad N(A) = \left\{ c \begin{bmatrix} 1 \\ -1 \end{bmatrix} : c \in \mathbb{R} \right\}$$

In general:

1) Suppose for some matrix A and some b

that x solves $Ax=b$.

For all $v \in N(A)$ $\quad A(x+v)=b$ as well.

$$\left(A(x+v) = Ax + Av = b + 0 = b \right).$$

2) Suppose x_1 and x_2 are solutions of $Ax=b$.

Then let $v = x_2 - x_1$.

Then $Av = A(x_2 - x_1) = Ax_2 - Ax_1$

$$= b - b = 0.$$

So $v \in N(A)$. So $x_2 = x_1 + v$

with $v \in N(A)$.

$Ax = 0$

That is: All solutions of $Ax = b$

are of the form $\hat{x} + v$

where $\hat{x}$ is one solution and

where $v \in N(A)$ is arbitrary.

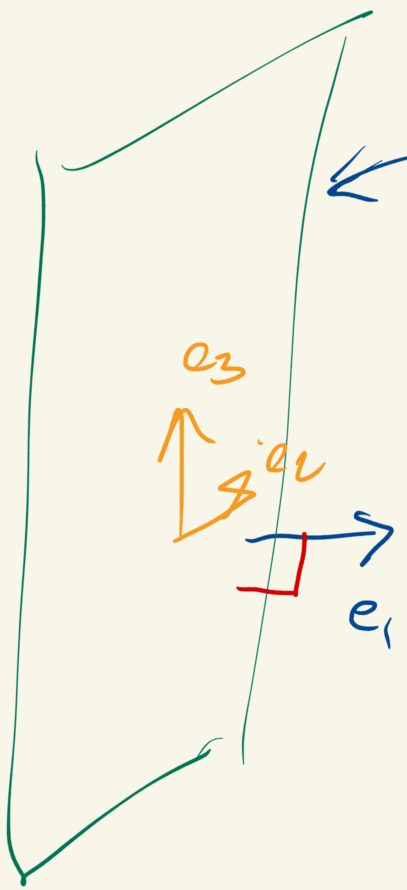
Some null spaces:

e_1

$$A = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \qquad N(A) = ?$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad x_1 = 0$$

$$N(A) = \left\{ \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$



$N(A)$

It's a plane.

$x = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ $Ax = 5$ $A = [1, 0, 0]$

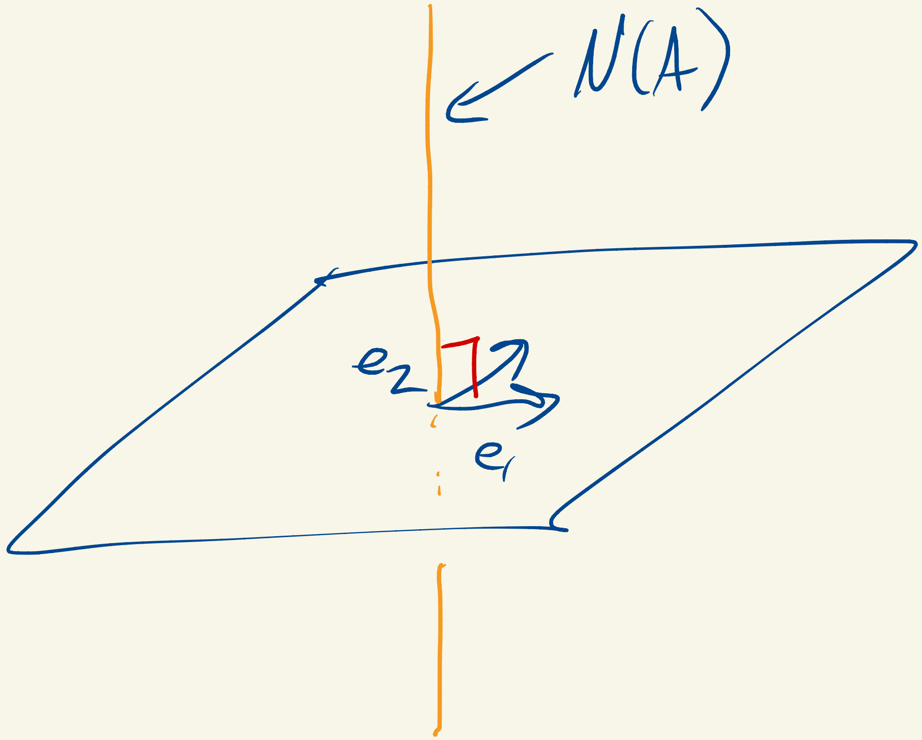
$x = \begin{bmatrix} 5 \\ x_2 \\ x_3 \end{bmatrix}$ $Ax = 5$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $N(A) = ?$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\underline{N(A) = \{(0, 0, x_3) : x_3 \in \mathbb{R}\}}$

$(0, 0, 1)$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Observations:

The null space of A is the set of vectors that are perpendicular to all the rows of A. (By def)

Structure: Suppose $v_1, v_2 \in N(A)$

Then 1) $v_1 + v_2 \in N(A)$

$$A(v_1+v_2) = Av_1 + Av_2 = 0 + 0 = 0$$

2) $cv_1 \in N(A)$ for all numbers c.

$$A(cv_1) = c(Av_1) = c0 = 0.$$

$$\alpha v_1 + \beta v_2 \in N(A)$$

A collection of vectors satisfyis 1) and 2) above

If we want to solve $Ax = b$

then $\hat{x} = A^{+}b$ is a solution.

But there are many solutions.

$\hat{x}$ has the smallest norm of all the

solutions of $Ax = b$.