$x, y \in \mathbb{R}^n$ $\quad x + y \quad$ costs n $\underbrace{\text{floating point operations}}_{\text{flops}}$

$c \in \mathbb{R}$ $\; x \in \mathbb{R}^n$ $\quad cx \quad$ n flops

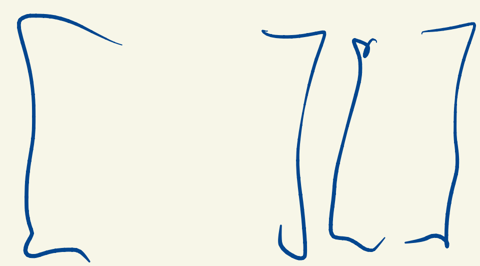
$x \cdot y \qquad x, y \in \mathbb{R}^3$

$x_1 y_1 + x_2 y_2 + x_3 y_3 \to 5$ flops

$x, y \in \mathbb{R}^n \qquad$ n mults, $n-1$ adds $\to$ $2n-1$ flops

$\overset{n \times n}{A} \;\; \overset{\in \mathbb{R}^n}{x}$

$n(2n-1)$ flops



$2n^2 - n$ flops

$\sim 2n^2$ flops

Solving $Rx = b$ R is <u>upper triangular</u>

$$r_{nn} x_n = b_n \qquad \text{1 flop}$$

$$r_{n-1\,n-1} x_{n-1} + r_{n-1\,n} x_n = b_{n-1} \qquad \text{1 mult, 1 sub, 1 division}$$

3 flops

$$r_{n-2\,n-2} x_{n-2} + \square\, x_{n-1} + \square\, x_n = b_{n-2}$$

$$x_{n-2} = \frac{1}{r_{n-2\,n-2}} \left[b_{n-2} - \square\, x_{n-1} - \square\, x_n \right]$$

2 mult, 2 sub, 1 division

5 flops

x_{n-3}

n: 1 flop

$n-1$: 3 flops

$n-2$: 5

$\vdots$

1 $2(n-1)+1$ flop

$$1 + 3 + 5 + 7 \cdots + (2(n-1)+1)$$

$$1 + (2+1) + (4+1) + \cdots + (2(n-1)+1)$$

$$n + 0 + 2 + 4 + \cdots + 2(n-1)$$

$$n + 2(0 + 1 + 2 + \cdots + n-1)$$

$$n + 2\sum_{k=1}^{n-1} k$$

$$1 + 2 + \cdots + 10$$
$$10 + 9 + \cdots + 1$$
$$\overline{11 + 11 + \cdots + 11}$$

$$n + 2\left(\frac{(n-1)\cdot n}{2}\right)$$

$$\frac{10 \cdot 11}{2}$$

$$n + (n-1)n$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$n^2$$

Solving $Rx = b$ takes n^2 floating point operations

This is cheap. Roughly the same as multiplying Ax

$$Ax = b$$

$$\downarrow$$

$$\underbrace{QR}x = b$$

given A and x

compute Ax $2n^2 - n$

$$Q^T Q R x = Q^T b$$

e_1

$$Rx = \underbrace{Q^T b}_{\to\, 2n^2 - n \,\sim\, 2n^2}$$

$$\underbrace{Rx = Q^T b}_{\to\, n^2}$$

total: $3n^2$

Compute A^{-1} once you have a QR fact.

$$A w_1 = e_1$$
$$A w_2 = e_2$$
$$\vdots$$
$$A w_n = e_n$$

solve these and then

$$A^{-1} = [w_1 \ w_2 \cdots w_n]$$

$$AA^{-1} = [Aw_1 \ Aw_2 \cdots Aw_n]$$
$$= [e_1 \ e_2 \cdots e_n]$$
$$= I$$

$3n^2$

Total is $\cancel{3n^3}$ n^3 operations to find A^{-1}

QR factorization is expensive order of $2n^3$ operations

$\downarrow$

$3n^3$ for the inverse

LU factorization

$$A = LU$$

L is lower triangular
U is upper triangular

$\frac{2}{3} n^3$

Want to solve $Ax = b$

$$LUx = b$$

1) Solve $Lw = b$ n^2

2) Solve $Ux = w$ n^2

$$LUx = Lw = b \qquad 2n^2 \text{ op.}$$

(Its cheaper)

Pseudo Inverse.

We've worked with solving $Ax=b$ when A is square and has linearly ind. columns.

(when it is invertible)

(QR factorization)

What if we relax the condition that A is square but still require that the columns of A are linearly independent.

If A has a left inverse then its columns are linearly independent

I'm going to show you how, when the columns are lin. ind to build a left inverse, A^{+} (pseudo inverse)

If you can find this left inverse and if there is a solution of $Ax=b$

then $$A^{+}Ax = A^{+}b$$

$$Ix = A^{+}b \Rightarrow x = A^{+}b.$$

(Can check afterwards if $Ax = b$)

Next chapter will explain why $A^+ b$ is even when there is no solution,

Claim: The columns of A are linearly independent if and only if the matrix

$$\underbrace{A^T A}_{\text{Gram matrix of } A} \quad \text{is invertible.}$$

A $m \times n$ where $m \geq n$

$A^T \quad n \times m \qquad A \quad m \times n \qquad A^T A \quad n \times n$

square!

Recall: The columns of a matrix B are linearly independent if and only if the only solution of $Bx = 0$ is $x = 0$.

Suppose A has linearly independent columns.

Suppose x is a vector where $A^T A x = 0$

Then $x^T A^T A x = 0$

$$(Ax)^T(Ax) = 0$$

$$\|Ax\|^2 = 0$$

so $\quad Ax = 0$

so $\quad x = 0$

Assumed A has lin. ind. cols.

If $\quad A^TAx = 0 \quad$ then $\quad x = 0.$

So $\quad A^TA$ has linearly ind. columns.

So A^TA is invertible.

Suppose the columns of A are not linearly ind.

Then there is a vector $x \neq 0$ with $Ax = 0$.

For this same x, $A^TAx = 0$

So $x \neq 0$ and $A^TAx = 0$.

So the columns of A^TA are not linearly independent.

So A^TA does not have an inverse.

If the columns of A are not linearly ind. then A^TA does not have an inverse.

So, if A^TA has an inverse,

then the columns of A are linearly independent.

$$Ax = 0$$

$$x = 0$$

A, columns are linearly independent

$$A^+ = (A^TA)^{-1}A^T$$

Moore-Penrose inverse

Pseudo inverse

Claim: A^+ is a left inverse of A.