Invertible Matrices

→ has an inverse ⇒ square

A , A^{-1}

Diagonal matrix $\operatorname{diag}(d_1, \ldots, d_n)$

$$D = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$$

claim: D has an inverse if and only if each $d_i \neq 0$.

Suppose each $d_i \neq 0$.

Claim $D^{-1} = \operatorname{diag}(d_1^{-1}, d_2^{-1}, \ldots, d_n^{-1})$

all ok because each $d_i \neq 0$.

$$D \cdot \begin{bmatrix} d_1^{-1} & & & 0 \\ & d_2^{-1} & & \\ & & \ddots & \\ 0 & & & d_n^{-1} \end{bmatrix} = \left[D\begin{bmatrix} d_1^{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \; D\begin{bmatrix} 0 \\ d_2^{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \; D\begin{bmatrix} 0 \\ \vdots \\ 0 \\ d_n^{-1} \end{bmatrix} \right]$$

$$= \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} = I$$

$$A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + \cdots + x_n a_n$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

Orthogonal matrix A

↳ square, columns are orthonormal

$AA^T = I$

↳ iff rows of A are o.n.

$A^TA = I \Leftrightarrow$ cols of A are o.n.

If A is square and $A^TA = I$

then A has a left inverse, A^T.

$A^{-1} = A^T$ $\qquad$ $AA^T = I$

Neat fact: For a square matrix, the columns are o.n.

if and only if the rows are o.n.

Remark: Consider A $m \times n$.

The columns of A are linearly independent if and only if the only solution of

$$Ax = 0 \quad \text{is} \quad x = 0.$$

$A = [a_1 \ a_2 \ \cdots \ a_n]$ The a_j's are linearly ind. iff the only linear combo $\underbrace{x_1 a_1 + \cdots x_n a_n}= 0$ has $x_1 = x_2 = \dots = x_n = 0$.

zero vector

$$A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

Then only x with $A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$ is $x = 0$.

If A is square then it is invertible if and only if the only solution of

$$Ax = 0 \quad \text{is} \quad x = 0.$$

"The columns of A are orthonormal"

"A is an orthogonal matrix"

↳ 1) square

2) columns are orthonormal

Upper triangular matrices:

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ 0 & 0 & r_{33} & \cdots & r_{3n} \\ & 0 & & \ddots & \\ & & & & r_{nn} \end{bmatrix}$$

when is R invertible?

precisely when each $r_{ii} \neq 0$.

what's the solution

$Rx = 0$

$r_{n-1\,n-1} x_{n-1} + r_{n-1\,n} x_n = 0$

$r_{nn} x_n = 0 \rightarrow x_n = 0$

$r_{n-1\,n-1} x_{n-1} = 0 \rightarrow x_{n-1} = 0$

not zero.

$x_1 = x_2 = \cdots = x_n = 0.$

What if some $r_{ii} = 0$

$$\begin{array}{cc} & \overbrace{\phantom{r_{11}\;r_{12}\;r_{13}}}^{3} \\ 2 & \left[\begin{array}{ccc|ccc} r_{11} & r_{12} & r_{13} & & & \\ 0 & r_{22} & r_{23} & & X & \\ \hline 0 & 0 & 0 & r_{44} & & \\ & & & & \ddots & \\ & 0 & & X & & \end{array}\right] \end{array}$$

Claim: the columns of this matrix are not linearly independent.

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \end{bmatrix}$$

three vectors in $\mathbb{R}^2$, not linearly independent

$$\begin{bmatrix} r_{1} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}$$

aren't linearly independent either

$$0\, x_k + * x_{k+1} \cdots + * x_n = b_k$$

R upper triangular, no diagonal entries are zero,

R has an inverse.

To solve $Rx = b$

do you find R^{-1} and then write $x = R^{-1} b$. ?

$$\begin{bmatrix} 6 & 5 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 2 \end{bmatrix}$$

$Ax = b$

$x = A^{-1}b$

R

"You do the inverse without finding the inverse matrix R^{-1}"

$x_3 = 2$

$4x_2 + 2 \cdot 2 = 0 \Rightarrow x_2 = -1$

$6x_1 - 5 + 6 = 7 \Rightarrow x_1 = 1$

How could we find R^{-1}?

$$R^{-1} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

$$RR^{-1} = I$$

$$RR^{-1} = \begin{bmatrix} Rv_1 & Rv_2 & Rv_3 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$$

$$\left.\begin{aligned} Rv_1 &= e_1 \\ Rv_2 &= e_2 \\ Rv_3 &= e_3 \end{aligned}\right\} \text{ Go solve these!}$$

$$R\,v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 6 & 5 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$6a + 5b + 3c = 0$$
$$4b + 2c = 0$$
$$c = 1$$

$$c = 1$$
$$4b = -2 \Rightarrow b = -1/2$$
$$6a = +\frac{5}{2} - 3$$
$$a = \frac{5}{12} - \frac{1}{2} = \frac{-1}{12}$$

$$\begin{bmatrix} -1/12 \\ -1/2 \\ 1 \end{bmatrix}$$

To find A^{-1}

"Go solve" $A v_k = e_k$ for vectors v_k.

Then $A^{-1} = [v_1 \ v_2 \cdots v_n]$

Our go-to technique for solving

$$Ax = b$$

when A is square and the colums of A are lin. ind. is QR factorization

$$A \to QR$$

upper triang

columns are ortho normal.

$$Ax = b$$

$$QRx = b$$

$$Q^T Q R x = Q^T b$$

$$\boxed{Rx = Q^T b}$$

solve this by back substitution.