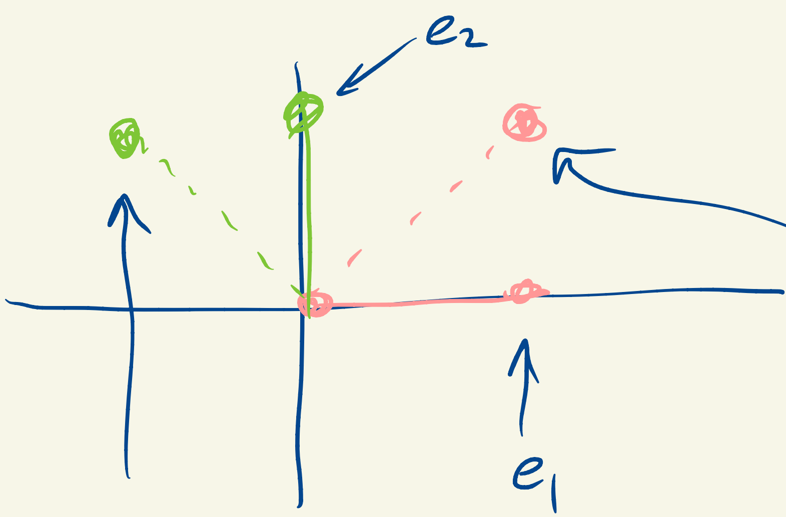
Matrix Powers

A $\quad A \cdot A = A^2$ only if A is square

$$\underset{n \times n}{A} \cdot \underset{n \times n}{A}$$

$$A \cdot A \cdot A = A^3$$

Function comp. with itself.



$$\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

A

$$e_1 \to \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$e_2 \to \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

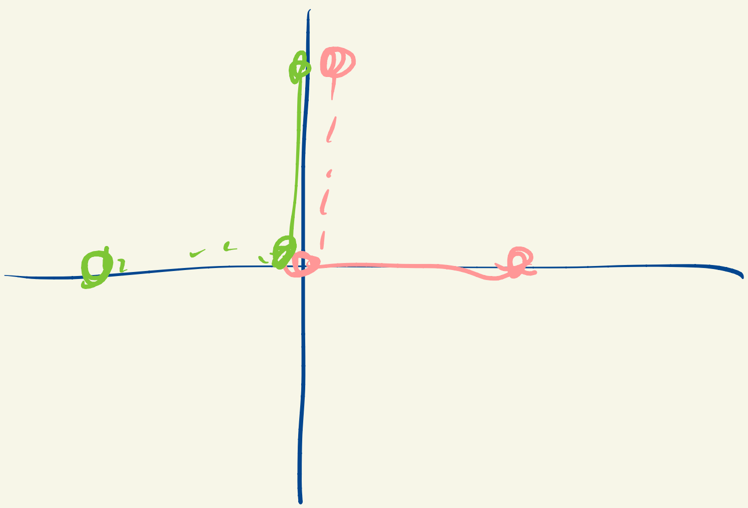
$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$\alpha = 1/\sqrt{2}$ $\qquad A = \begin{bmatrix} \alpha & -\alpha \\ \alpha & \alpha \end{bmatrix}$ $\qquad \alpha^2 = \frac{1}{2}$

$$A^2 = \begin{bmatrix} \alpha & -\alpha \\ \alpha & \alpha \end{bmatrix} \begin{bmatrix} \alpha & -\alpha \\ \alpha & \alpha \end{bmatrix} = \begin{bmatrix} 0 & -2\alpha^2 \\ 2\alpha^2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



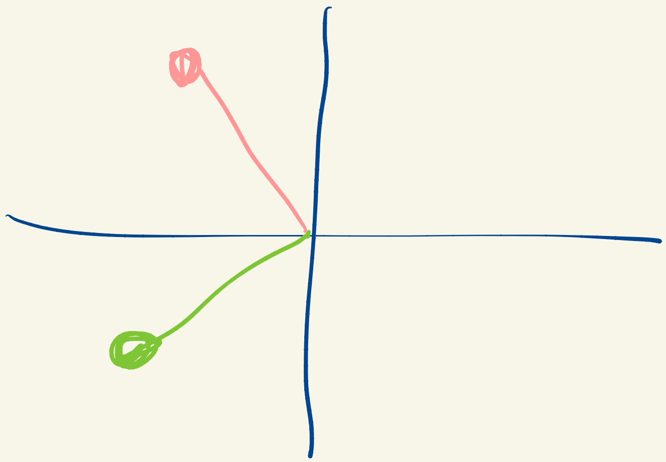
$e_1 \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad e_2 \to \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

i

A^3

$$A^3 = A^2 A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & -\alpha \\ \alpha & \alpha \end{bmatrix} = \begin{bmatrix} -\alpha & -\alpha \\ \alpha & -\alpha \end{bmatrix}$$



$$i^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= -I$$

$$a + jb$$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = aI + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Orthogonal Matrices and QR Factorization

A ~~suppose~~ the cols of A are orthonormal

A: $m \times k$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix}$$

$$a_i^T a_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$a_i^T a_i = \|a_i\|^2$$

$$\underset{k \times m}{A^T} \; \underset{m \times k}{A} = \underset{k \times k}{I}$$

$$(A^T A)_{ij} = a_i^T a_j = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

We say A is <u>orthogonal</u> if it is square and its columns are orthonormal.

$$\underset{\substack{\uparrow \\ n\times n}}{A^T}\,\underset{\substack{\uparrow \\ n\times n}}{A} = \underset{\substack{\uparrow \\ n\times n}}{I}$$

$$\cancel{A^TA = A^2}$$

e.g. identity matrix. square. $I^TI = II = I$

e.g. permutation matrices

square
each row has one 1
each column has one 1
all other entries are zero.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \end{bmatrix}$$

orthogonal

$c^2 + s^2 = 1$

2×2 rotation matrices

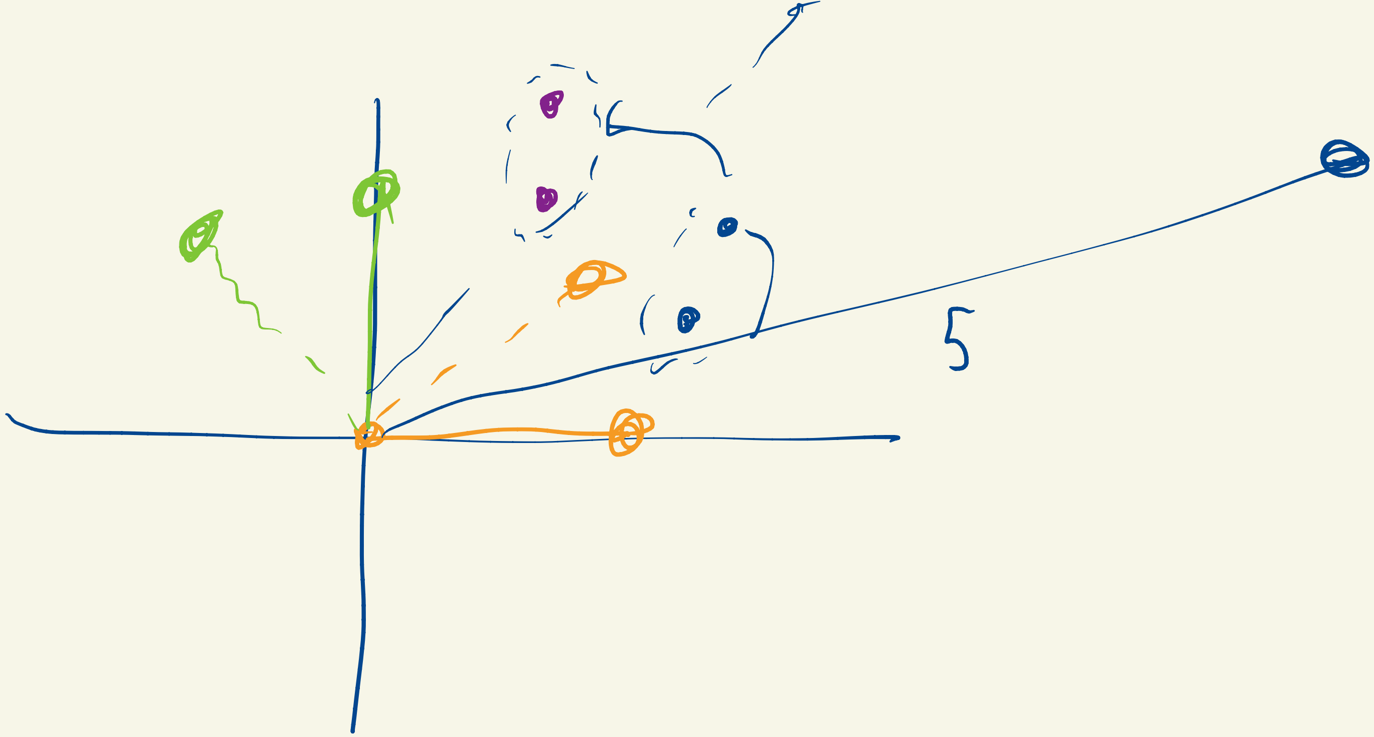
R

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

orthogonal

$R^T R = I$

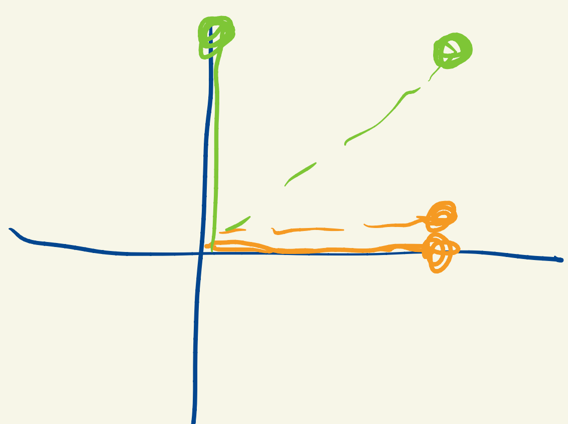
$R^2 \Rightarrow$ rotation by 2θ, not I



$$e_1 \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$e_2 \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



If A has orthonormal columns then

$$\|Ax\| = \|x\|$$

$$\angle(Ax, Ay) = \angle(x, y)$$

$$(Ax)^T(Ay) = x^Ty$$

next class

QR