$$(b_1, b_2, b_3) \leftarrow b$$

$$(a_3, a_2, a_1)$$

$$\uparrow a$$

$$a * b$$

"convolution of a with b"

$$(a_1b_1,\ a_1b_2+a_2b_1,\ a_3b_1+a_2b_3+a_1b_3,\ a_2b_3+a_3b_2,\ a_3b_3)$$

$a \in \mathbb{R}^m$

$b \in \mathbb{R}^n$

$n + m - 1$

---

Linear ~~Functions~~ Equations

Chapter 8

$$f(x) = c^T x$$

$f: \mathbb{R}^n \to \mathbb{R}$

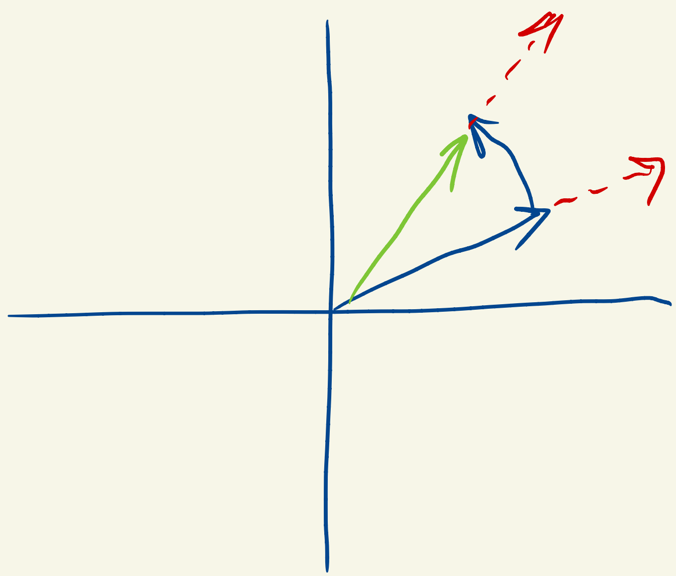
for all $x, y \in \mathbb{R}^n$ and all $c \in \mathbb{R}$

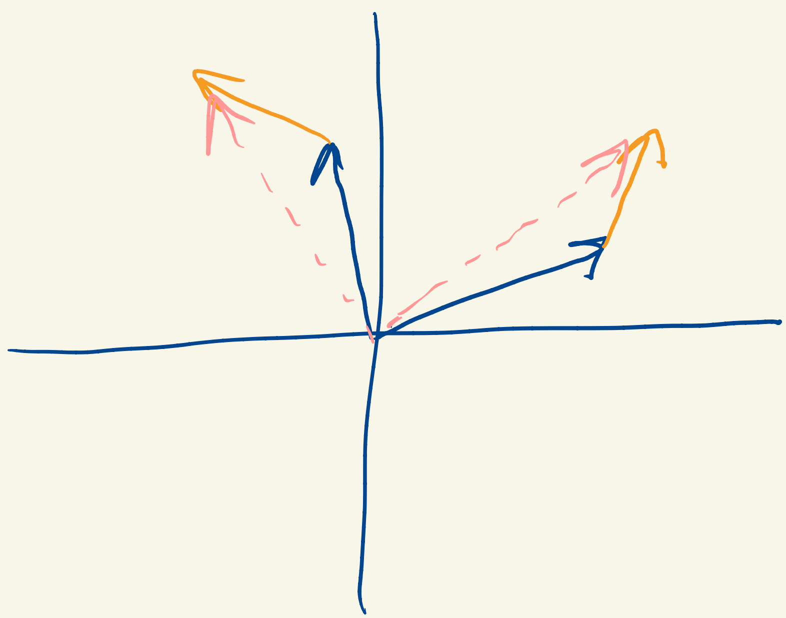
$$\left.\begin{aligned} f(x+y) &= f(x) + f(y) \\ f(cx) &= c f(x) \end{aligned}\right] \text{ linearity}$$

$f: \mathbb{R}^n \to \mathbb{R}^m$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \leftarrow \text{superposition}$$

Rotation





Given an $m \times n$ matrix $A$ we'll define a function (i.e. a map)

$$f_A(x) \to Ax \qquad f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

Claim: $f_A$ is linear.

$$\begin{aligned} f_A(x+z) &= A(x+z) \\ &= Ax + Az \\ &= f_A(x) + f_A(z) \end{aligned}$$

$$\begin{aligned} f_A(cx) &= A(cx) \\ &= cAx \\ &= c f_A(x) \end{aligned}$$

Claim: If $f: \mathbb{R}^n \to \mathbb{R}^m$ is linear then there is an $m \times n$ matrix $A$ such that

$$f(x) = f_A(x) = Ax \quad \text{for all } x \in \mathbb{R}^n$$

$\boxed{\text{"Linear maps can be represented by matrix vector multiplication"}}$

$$f: \mathbb{R}^n \to \mathbb{R}^m$$

$$\overbrace{e_1, \dots, e_n}$$

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$A = \overbrace{\begin{bmatrix} f(e_1), & f(e_2), & \dots, & f(e_n) \end{bmatrix}}^{n} \Big]\, m$$

$$x = (x_1, x_2, \ldots, x_n)$$

$$f(x) = f(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$$

$$= x_1 f(e_1) + x_2 f(e_2) + \cdots + x_n f(e_n)$$

$$= \begin{bmatrix} f(e_1) & f(e_2) & \cdots & f(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

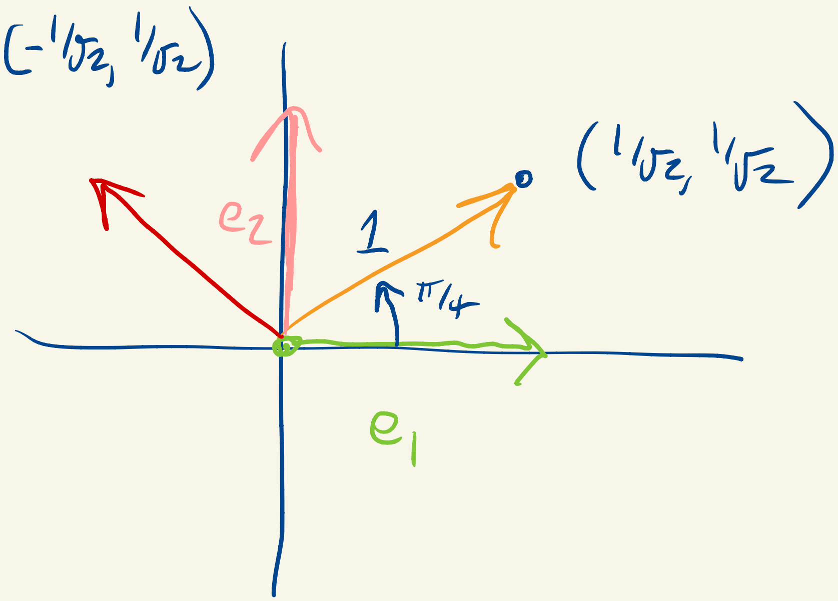
$$= A x$$

$$= f_A(x)$$

$$R_{\frac{\pi}{4}} = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$e_1$ goes here.

$e_2$ goes here.




$$R_{\frac{\pi}{4}}(e_1) = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= 1 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + 0 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Examplos

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \longrightarrow \begin{bmatrix} x_n \\ x_2 \\ \vdots \\ x_{n-1} \\ x_1 \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \longrightarrow \begin{bmatrix} x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_1 \end{bmatrix} \qquad e_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$e_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} \quad Ax = \begin{bmatrix} 0 \cdot x_1 + 0 x_2 \cdots + 0 x_4 + 1 \cdot x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad = \begin{bmatrix} x_5 \\ x_2 \\ x_3 \\ x_4 \\ x_1 \end{bmatrix}$$

$$X \longrightarrow \begin{bmatrix} x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

"see text for general permutation matrices"

$$(13.6,\ 9,\ -11) \qquad (-11,\ 9,\ 13.6)$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} x_1 \\ 2x_2 \\ 3x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \to \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 \\ x_1 + x_2 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 + x_4 \\ x_1 + x_2 + x_3 + x_4 + x_5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$y \longrightarrow y - \text{avg}(y)\vec{1}$$

"Demeaning"

removes the mean

$$\mathbb{R}^5 \to \mathbb{R}^5$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -1/5 \\ -1/5 \\ -1/5 \\ -1/5 \end{bmatrix}$$

$$e_2 \longrightarrow \begin{bmatrix} -1/5 \\ 4/5 \\ -1/5 \\ -1/5 \\ -1/5 \end{bmatrix}$$

$$A = \begin{bmatrix} 4/5 & -1/5 & -1/5 & \cdots & -1/5 \\ -1/5 & 4/5 & & & \vdots \\ -1/5 & -1/5 & 4/5 & & -1/5 \\ -1/5 & -1/5 & & 4/5 & -1/5 \\ -1/5 & \cdots & \cdots & -1/5 & 4/5 \end{bmatrix}$$

$$\begin{bmatrix} 1-1/n & -1/n & \cdots & -1/n \\ -1/n & 1-1/n & -1/n \cdots & -1/n \\ & \vdots & & \\ -1/n & \cdots & -1/n & 1-1/n \end{bmatrix}$$