The collection is orthonormal if in addition

$\|a_j\| = 1$ for all j.

$a_i^T a_i = \|a_i\|^2$

$$a_i^T a_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\begin{aligned} 3x - 2y + 4z &= 8 \\ -x + 4y + 2z &= 5 \\ 8x + 7y + z &= 9 \end{aligned}$$

"Find x, y, z solving this"

$$x \underbrace{\begin{bmatrix} 3 \\ -1 \\ 8 \end{bmatrix}}_{v_1} + y \underbrace{\begin{bmatrix} -2 \\ 4 \\ 7 \end{bmatrix}}_{v_2} + z \underbrace{\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}}_{v_3} = \underbrace{\begin{bmatrix} 8 \\ 5 \\ 9 \end{bmatrix}}_{b}$$

"Find a linear combination of v_1, v_2, v_3 that equals b"

Basis for $\mathbb{R}^3$

3 vectors in $\mathbb{R}^3$ that are linearly independent

We showed that if v_1, v_2, v_3 are a basis for $\mathbb{R}^3$ then given any x in $\mathbb{R}^3$ there are $\beta_1, \beta_2, \beta_3$ with

$$\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = x$$

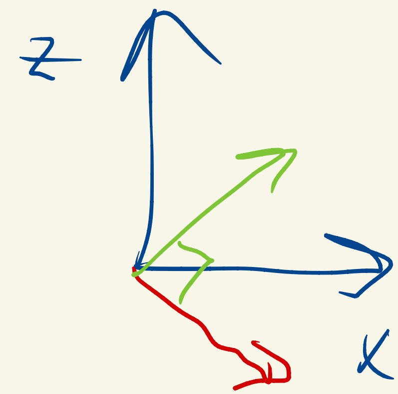
Moreover, because of linear independence there is only one way to do this

$$a_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \quad a_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad a_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$a_1^T a_2 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 0$$

$$\|a_1\|^2 = a_1^T a_1 = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 0 \cdot 0 + \left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)$$

$$= \frac{1}{2} + 0 + \frac{1}{2} = 1$$



Claim: A orthonormal collection of vectors is always linearly independent.

Suppose $a_1, \dots, a_k$ are orthonormal.

Suppose $\beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_k a_k = 0$.

Job: Show every $\beta_j = 0$.

Observe

$$a_1^T(\beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_k a_k) = a_1^T 0 = 0$$

But

$$a_1^T(\beta_1 a_1 + \cdots + \beta_k a_k) =$$

$$\beta_1 a_1^T a_1 + \beta_2 a_1^T a_2 + \cdots + \beta_k a_1^T a_k$$

$$= \beta_1 \cdot 1 + 0 + \cdots + 0$$

$$= \beta_1$$

Repeat for $a_2, \ldots, a_k$ to conclude $\beta_2 = 0, \ldots, \beta_k = 0$

Now suppose $a_1, \dots, a_n$ are orthonormal in $\mathbb{R}^n$

It's a linearly independent collection of n vectors in $\mathbb{R}^n$ and therefore is a basis for $\mathbb{R}^n$

Given any $x \in \mathbb{R}^n$ we can write x uniquely in the form

$$x = \beta_1 a_1 + \beta_2 a_2 + \dots + \beta_n a_n$$

How do you determine the β_j's?

Observe
$$\begin{aligned} a_1^T x &= a_1^T(\beta_1 a_1 + \dots + \beta_n a_n) \\ &= \beta_1 a_1^T a_1 + \beta_2 a_1^T a_2 + \dots + \beta_n a_1^T a_n \\ &= \beta_1 \cdot 1 + \beta_2 \cdot 0 + \dots + \beta_n \cdot 0 \end{aligned}$$

$= \beta_1$

So: If you know x and the orthonormal basis $a_1, \ldots, a_n$ you can compute

$$\beta_k = a_k^T x$$

Then $x = \beta_1 a_1 + \cdots + \beta_n a_n$

$$a_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \quad a_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad a_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Lets find $\beta_1, \dots, \beta_3$ with $x = \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3$

$$\beta_1 = a_1^T x = \frac{1}{\sqrt{2}} + 0 + \frac{2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$\beta_2 = a_2^T x = -\frac{1}{\sqrt{2}}$$

$$\beta_3 = a_3^T x = 2$$

$$\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = \frac{3}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

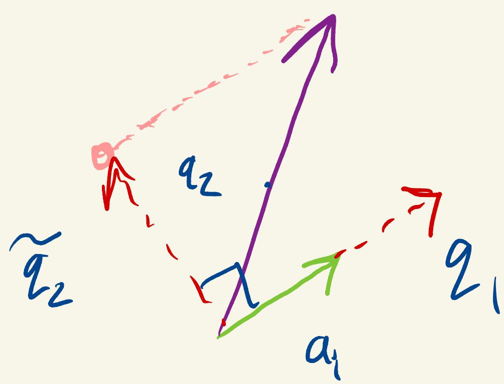
$$= \begin{bmatrix} 3/2 \\ 0 \\ -3/2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/2 \\ 2 \\ -4/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = x$$ ☺

Gram Schmidt procedure

Start with arbitrary $a_1, \ldots, a_k$

and make a <u>related</u> $q_1, \ldots, q_k$ who are orthonormal.



$$q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|}$$