Hence a_1, a_2, a_3 are linearly independent.

A reason to care about linear independence:

Suppose $a_1, \dots, a_k$ are linearly independent.

Suppose $x = \beta_1 a_1 + \cdots + \beta_k a_k$

Suppose also $x = \hat{\beta}_1 a_1 + \cdots + \hat{\beta}_k a_k$

Claim: $\beta_j = \hat{\beta}_j$ for all j.

$$0 = (\beta_1 - \hat{\beta}_1) a_1 + \cdots + (\beta_k - \hat{\beta}_k) a_k$$

Each coefficient must be 0 because the a_j's are linearly independent.

So $\beta_1 - \hat{\beta}_1 = 0$ i.e. $\beta_1 = \hat{\beta}_1$

$$\beta_j = \hat{\beta}_j \quad j = 1, \dots, k.$$

$a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Claim: any x in $\mathbb{R}^n$ is a linear combo of a and b.

$x = (x_1, x_2)$ $\quad x \overset{?}{=} \alpha a + \beta b = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} \alpha + \beta \\ \beta \end{pmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha + \beta \\ \beta \end{bmatrix}$$

$\Rightarrow \beta = x_2$ $\qquad$ $x_1 = \alpha + \beta$
$\qquad = \alpha + x_2$ $\Rightarrow \alpha = x_1 - x_2$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are linearly independent.

$$\beta_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} \beta_1 + \beta_2 \\ \beta_2 \end{bmatrix} = 0 \quad \Rightarrow \beta_2 = 0 \quad \Rightarrow \beta_1 = 0$$

A collection $a_1, \ldots, a_k$ of vectors in $\mathbb{R}^n$
is called a <u>basis</u> for $\mathbb{R}^n$ if

1) The collection is linearly independent

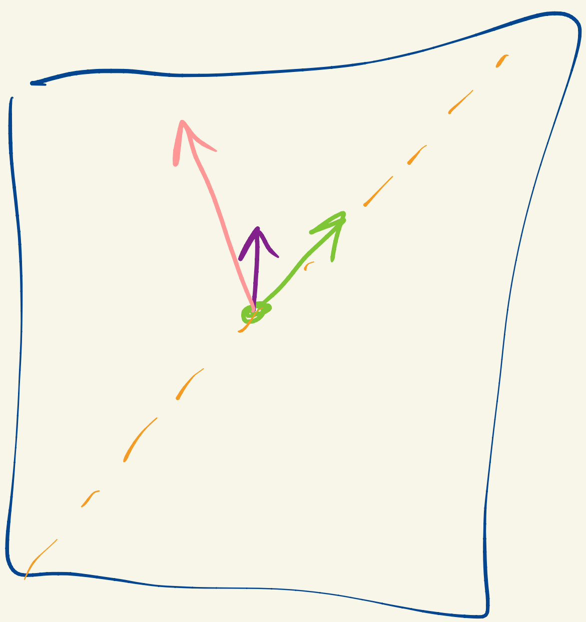
Not too big

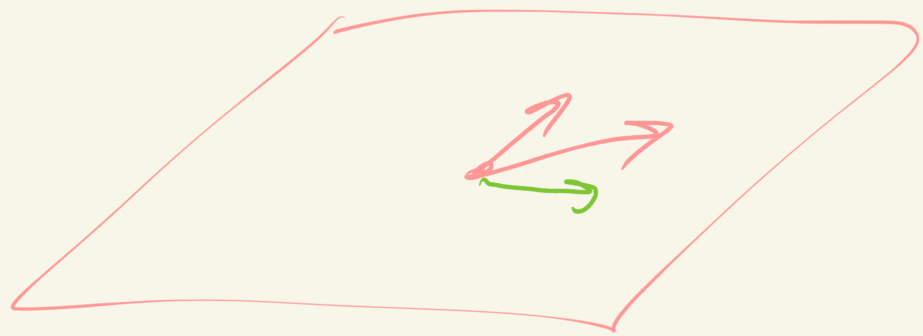
2) Every vector x in $\mathbb{R}^n$ can be written as a linear combo of the a_j's:

$$x = \beta_1 a_1 + \cdots + \beta_k a_k$$

for some β_j's.

Not too small





If you have a basis $a_1, \dots, a_k$ for $\mathbb{R}^n$

then every vector in $\mathbb{R}^n$ can be written as

a unique linear combination of the a_j's.

$$a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ form a basis for $\mathbb{R}^2$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ forms a basis for $\mathbb{R}^2$

Facts:

A) If $a_1, \dots, a_k$ are linearly independent in $\mathbb{R}^n$ then $k \le n$.

B) If $a_1, \dots, a_k$ are vectors in $\mathbb{R}^n$ and $k < n$ then there is a vector $x \in \mathbb{R}^n$ that is not a linear combo of the a_k's.

Consequence: Every basis for $\mathbb{R}^n$ has n vectors

Consequence ~~Observation~~: Suppose $a_1, \dots, a_n$ are linearly independent vectors in $\mathbb{R}^n$. Then they form a basis for $\mathbb{R}^n$.

Let $x \in \mathbb{R}^n$.

The collection $a_1, \dots, a_n, x$ has $n+1$ vectors in $\mathbb{R}^n$. So it is linearly dependent.

So there are coefficients $\beta_1, \dots, \beta_{n+1}$

$$\beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_n a_n + \beta_{n+1} x = 0$$

with not all β_j's $= 0$.

Because the a_j's are linearly independent β_{n+1} must be non zero.

Then $x = -\frac{\beta_1}{\beta_{n+1}} a_1 - \cdots - \frac{\beta_n}{\beta_{n+1}} a_n$.

So x is a linear combination of the a_j's.

$$\boxed{\text{n linearly independent vectors in } \mathbb{R}^n \text{ are a basis for } \mathbb{R}^n}$$

E.g: $a_1 = \begin{bmatrix} 1.2 \\ -2.6 \end{bmatrix}$ $a_2 = \begin{bmatrix} -0.3 \\ -3.7 \end{bmatrix}$

To show these form a basis I need to show they are linearly independent.

$\beta_1 a_1 + \beta_2 a_2 = 0$ job: $\beta_1 = \beta_2 = 0$

$\beta_1 \cdot 1.2 + \beta_2 (-0.3) = 0$ Exercise: $\beta_1 = \beta_2 = 0$.

$\beta_1 \cdot (-2.6) + \beta_2 (-3.7) = 0$

$$a_1 = \begin{bmatrix} 1.2 \\ -2.6 \end{bmatrix} \qquad a_2 = \begin{bmatrix} -0.3 \\ -3.7 \end{bmatrix}$$

$$x = \begin{bmatrix} 4 \\ -7 \end{bmatrix} \qquad x = \beta_1 \begin{bmatrix} 1.2 \\ -2.6 \end{bmatrix} + \beta_2 \begin{bmatrix} -0.3 \\ -3.7 \end{bmatrix}$$

for some β_1, β_2

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We say a collection of vectors $a_1, \ldots, a_k$ is orthogonal

if $a_j^T a_i = 0$ if $j \neq i$. (The vectors are mutually perpendicular)

The collection is orthonormal if in addition

$$\|a_j\| = 1 \text{ for all } j.$$

$$a_i^T a_i = \|a_i\|^2$$

$$a_i^T a_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$