$f : \mathbb{R}^n \to \mathbb{R}$

f is a linear function

1) additivity $\quad f(x+y) = f(x) + f(y)$

2) $\quad f(cx) = c\,f(x)$

superposition $\quad f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$x = (1,3) \qquad \alpha = 5$

$y = (-2,6) \qquad \beta = -2$

$$f\big(\underbrace{5(1,3) - 2(-2,6)}_{f(9,3)}\big) = 5\,f(1,3) - 2\,f(-2,6)$$

$f(1,3) = 19$

$f(5,15) = f(5(1,3)) = 5f(1,3) = 5 \cdot 19$

$5 \cdot 1, 5 \cdot 3$

There exists a vector c such that

$f(x) = c^T x$ $\qquad$ $f: \mathbb{R}^s \to \mathbb{R}$

$f(x) = \underbrace{45}_{\uparrow} \cdot x$

Affine:

$f(x) = c^T x + v$ number maybe $\neq 0$

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

iff $\alpha + \beta = 1$

$f(\alpha x + \beta y + \gamma z)$

$= \alpha f(x) + 1 f(\beta y + \gamma z)$

$= \alpha f(x) + \beta f(y) + \gamma f(z)$

$f(\alpha x + \beta y + \gamma z) = \alpha f(x) + \beta f(y) + \gamma f(z)$

iff $\alpha + \beta + \gamma = 1$

$\hookrightarrow$ affine

$f : \mathbb{R}^n \to \mathbb{R}$, linear

$f : \mathbb{R}^m \to \mathbb{R}^n$ linear

$\hookrightarrow$ understand linear functions like this

What is the inner structure of these maps?

Given some $y \in \mathbb{R}^n$ can you solve $f(x) = y$?

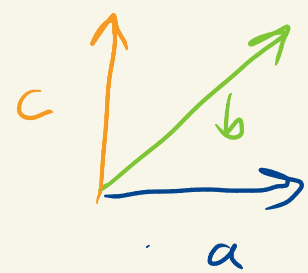
If you can't what's a best ~~p~~ossible almost solution?

Applications (ODEs, Markov chains)

$$f(x) = y$$

$\mathbb{R}^2$

$$a = (1,0)$$
$$b = (1,1)$$
$$c = (0,1)$$



Any vector in $\mathbb{R}^2$ can be written in the form

$$\alpha a + \beta b + \gamma c$$

$$(5,7) = 5 \cdot b + 2c = (5,5) + (0,2) = (5,7)$$

$$(5,7) = 5a + 7c$$

$$0 = \underbrace{-5a + 5b - 5c}$$

→ "redundancy"

too many vectors to be a efficient description of others

Def: Let $a_1, a_2, \dots, a_k$ be vectors in $\mathbb{R}^n$.

The collection is <u>linearly dependent</u> if there exist numbers $\beta_1, \beta_2, \dots, \beta_k$ not all zero such that

$$\beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_k a_k = 0.$$

So: $a = (1,0)$, $b = (1,1)$, $c = (0,1)$ are linearly dependent.

$$a - b + c = 0$$

$$c = b - a$$

$$b = a + c$$

$$a = c - b$$

A collection $a_1, \ldots, a_k$ is linearly dependent iff one of the a_j's can be written as a linear combination of the others.

$$\beta_1 a_1 + \cdots + \beta_k a_k = 0$$

If $\beta_1 \neq 0$ $\qquad a_1 = -\frac{1}{\beta_1}\left[\beta_2 a_2 + \beta_3 a_3 + \cdots + \beta_k a_k\right]$

A collection $a_1, \dots, a_k$ is linearly independent
if it is not linearly dependent.

lindep.

There exist $\beta_1, \dots, \beta_k$, not all zero, with $\beta_1 a_1 \cdots + \beta_k a_k = 0$

linearly independent: if you have numbers

$$\beta_1 a_1 + \beta_2 a_2 + \cdots + \beta_k a_k = 0$$

then $\beta_1 = \beta_2 = \cdots = \beta_k = 0$.

$a_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad a_3 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

Show that a_1, a_2, a_3 are linearly dependent.

Job: a) find $\beta_1, \beta_2, \beta_3$, not all zero, $\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = 0$

or write one of the vectors as a linear combo

b) of the others

a) $\beta_1 = 1, \ \beta_2 = 1, \ \beta_3 = -1$

$$a_1 + a_2 - a_3 = 0$$

b) $a_3 = a_1 + a_2$

$$a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad a_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \qquad a_1 + a_2 = 0$$

$$a_1 = -a_2$$

$a_1 = \cdots \qquad a_2 = \cdots \qquad a_3 = 0$

$$0 \cdot a_1 + 0 \cdot a_2 + 1 a_3 = 0$$

$$a_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad a_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Claim these are linearly independent.

Need to show that if

$$\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = 0 \quad \text{then} \quad \beta_1 = \beta_2 = \beta_3 = 0$$

$$\beta_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \beta_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \beta_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} \beta_1 + \beta_2 + \beta_3 \\ -\beta_1 + \beta_3 \\ \beta_1 + \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \beta_1 + \beta_2 + \beta_3 &= 0 & &\Rightarrow \beta_3 = 0 \\ -\beta_1 + \beta_3 &= 0 & &\Rightarrow \beta_1 = 0 \\ \beta_1 + \beta_2 &= 0 & &\Rightarrow \beta_2 = 0 \end{aligned}$$

Hence a_1, a_2, a_3 are linearly <u>independent</u>.