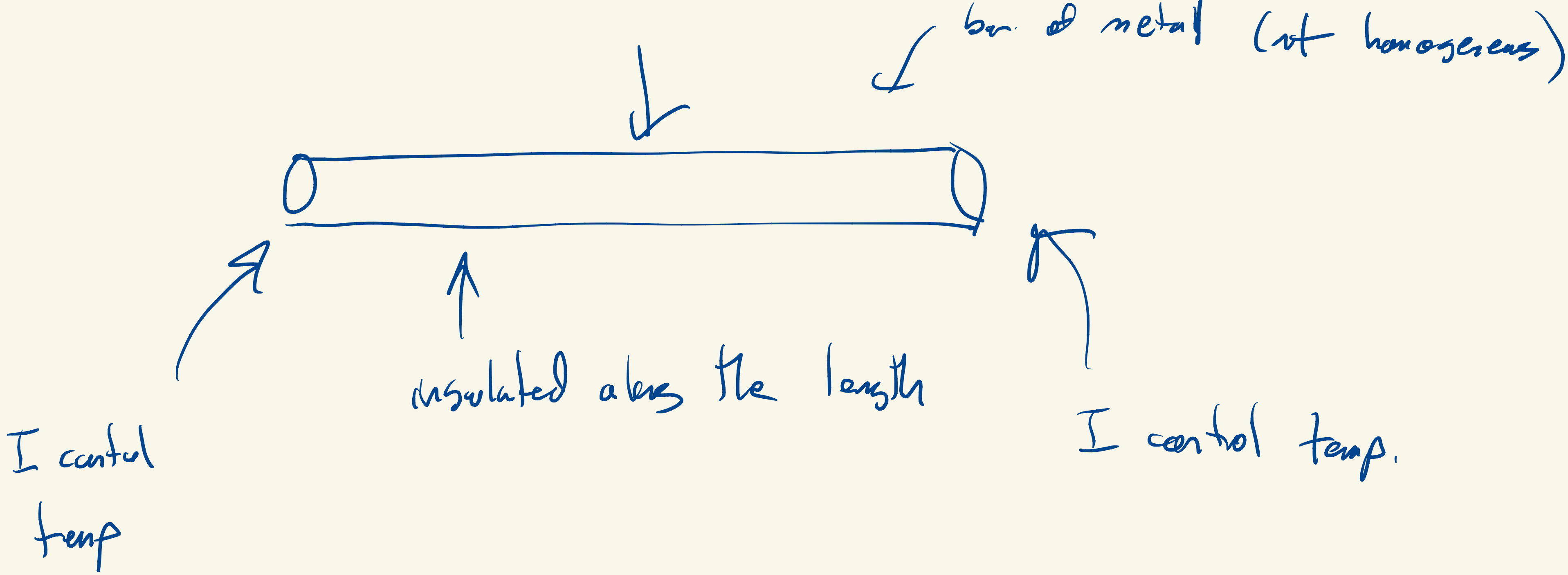
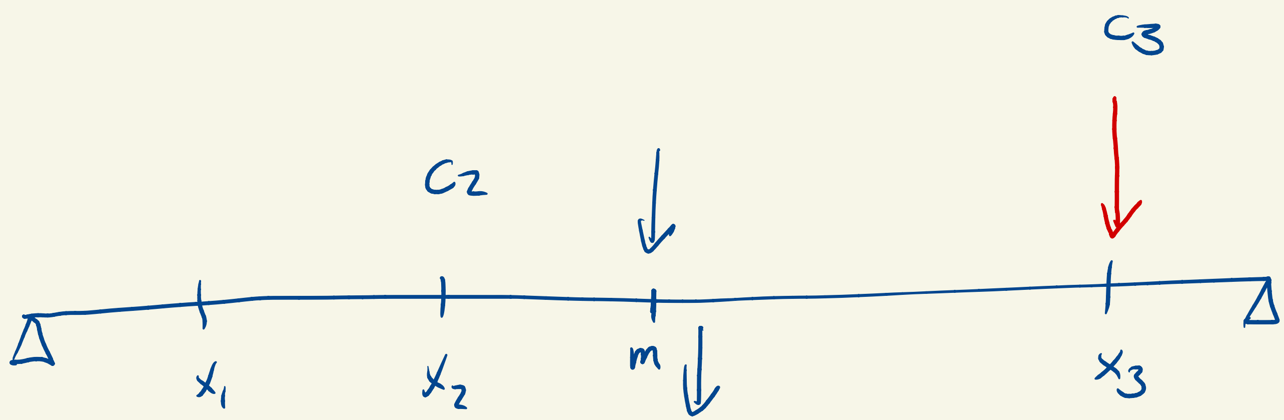


c_1, c_2, c_3



Want to measure the sag of the beam at m

as a response to loads at x_1, x_2, x_3

Say s is measured in mm.

Forces will be encoded as masses. (metric tonnes)

Masses: m_1, m_2, m_3

s sag

$$s(m_1, m_2, m_3) = \underbrace{c_1 m_1 + c_2 m_2 + c_3 m_3}$$

This is a linear function.

$c^T m$

$c = (c_1, c_2, c_3)$

$m = (m_1, m_2, m_3)$

where the coefficients c_i are known as sensitivities and have units of mm/tonne

$$f(x+y) = f(x) + f(y)$$

$$f(\alpha x) = \alpha f(x)$$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$ "superposition"

$\alpha, \beta \in \mathbb{R}$

Claim: every linear function $\underbrace{f : \mathbb{R}^n \to \mathbb{R}}$ can be written in the form "f is a function from $\mathbb{R}^n$ to $\mathbb{R}$"

$$f(x) = c^T x \quad \text{for some fixed vector } c.$$

$$e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$$

$f \leftarrow$ we know it is linear.

$c_k = f(e_k) \quad k = 1, \ldots, n$

(all these vectors have length n)

$$X = (x_1, x_2, \ldots, x_n)$$

$$= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

$$\big(= x_1 (1, 0, \ldots, 0) + x_2 (0, 1, 0, \ldots, 0) + \cdots + x_n (0, 0, \ldots, 0, 1) \big)$$

$$\big(= (x_1, 0, \ldots, 0) + (0, x_2, 0, \ldots, 0) + \cdots + (0, 0, \ldots, 0, x_n) \big)$$

$$\big(= (x_1, x_2, \ldots, x_n) \big)$$

$$= X$$

$$(e_3)_4 = 0$$

$$X = (x_1, x_2, \dots, x_n)$$

$$= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

$$f(x) = f(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$$

$$= f(x_1 e_1) + f(x_2 e_2) + \cdots + f(x_n e_n))$$

$$= x_1 \underbrace{f(e_1)} + x_2 f(e_2) + \cdots + x_n f(e_n))$$

$$= x_1 c_1 + x_2 c_2 + \cdots + x_n c_n$$

$$= c^T X$$

$y = mx + b$

$f(x) = 3x - 7$

$y = 3x - 7$

$f : \mathbb{R} \to \mathbb{R}$

$f(1) = -4$

$f(0) = -7$

$f(x+w) = f(x) + f(w)$

$f(\alpha x) = \alpha f(x)$

$$\underbrace{f(0+1)}_{\substack{f(1) \\ -4}} = \underbrace{f(0) + f(1)}_{-11}$$

f ← linear

$$f(0) = f(0+0)$$
$$= f(0) + f(0)$$

$$f(0) = 2f(0)$$
$$0 = f(0)$$

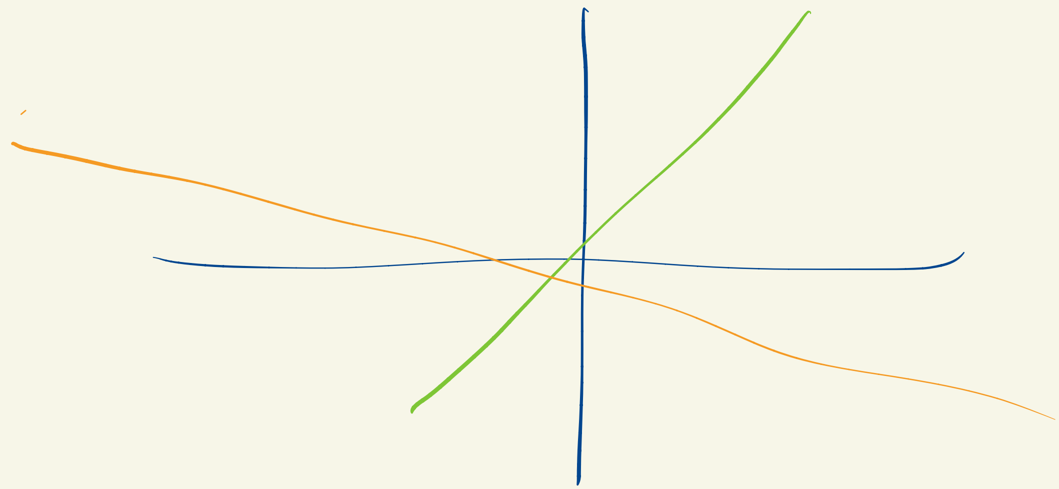
If f is linear then $f(0) = 0$.

$f(x) = mx + b$

$f(0) = b$

$\underbrace{f(x) = mx}$

This is linear.



A function $f(x)$ of the form

$$f(x) = c^T x + b \quad \text{where } b \in \mathbb{R}$$

is called an <u>affine</u> function.

Linear functions: $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

Affine functions $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

but only if $\alpha + \beta = 1$.

(limited superposition)

↳ see text.

$$f(\overbrace{x_1, x_2}^{x=(x_1,x_2)}) = x_1 \cdot e^{x_2}$$

$$f(3,0) = 3 \cdot e^0 = 3$$

$f(3.1, -0.2)$ is close to 3

Can we approximate this value using only the skills of 4^{th} grade?

$$\frac{\partial f}{\partial x_1} = e^{x_2} \qquad \frac{\partial f}{\partial x_2} = x_1 e^{x_2}$$

$$\nabla f = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix}$$

at $(3,0)$

$$\nabla f = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

We can approximate $f(x_1, x_2)$ for (x_1, x_2) near $(3,0)$ as follows:

$$\hat{f}(x_1, x_2) = \underbrace{3}_{f(3,0)} + \overset{\frac{\partial f}{\partial x_1} \text{ at } (3,0)}{1}\left(x_1 - \overset{(3,0)}{3}\right) + \underbrace{3}_{\frac{\partial f}{\partial x_2} \text{ at } (3,0)}\left(x_2 - \overset{(3,0)}{0}\right)$$

$$\hat{f}(3.2, -0.1) = 3 + 1(0.2) + 3(-0.1)$$

$$= 3 + 0.2 - 0.3$$

$$= 3 - 0.1 = 2.9$$

$$f(3.2, -0.1) = 2.895$$

$$f(1, -10) = 4.5 \times 10^{-5} \quad (\text{close to } 0)$$

$$\hat{f}(1, -10) = 31$$

$$w = (3, 0)$$

$$\hat{f}(x) = f(3,0) + \underbrace{(\nabla f)}_{\text{evaluated at } (3,0)}^{T} \left(x - (3,0)\right)$$

$$x = (x_1, x_2)$$

linear approximation

$$\hat{f}(x) = f(w) + (\nabla f)^{T} (x - w)$$

evaluated at w

an approximation of f near w. $\nabla f(w)$

$$\hat{f}(x) = \underbrace{\left[f(w) - \nabla f^T w\right]}_{b} + (\nabla f)^T x$$

This is an affine function of x