$$a^T \cdot b = a_1 b_1 + a_2 b_2 + \dots + a_n \cdot b_n$$

$$= b_1 a_1 + b_2 \cdot a_2 + \dots + b_n \cdot a_n$$

$$= b^T a$$

$$a = (a_1, a_2, \dots, a_{n+1})$$

t is a number

$$b = \underbrace{(1, t, t^2, \dots, t^n)}_{n+1}$$

$$a^T b = \underbrace{a_1 + a_2 t + a_3 t^2 + \dots + a_{n+1} t^n}_{\text{polynomial!}}$$

$$a^T b = b^T a$$

$$\underline{(a+b)^T c = a^T c + b^T c} \leftarrow \text{exercise}$$

$$a^T(b+c) = a^T b + a^T c$$

$$a^T(\beta b) = \beta(a^T b)$$

$\beta \in \mathbb{R}$

"is a"

"in"

"is an element of"

$$(\beta a)^T b = \beta(a^T b)$$

$$(a+b)^T(c+d) = a^T(c+d) + b^T(c+d)$$

$$= a^T c + a^T d + b^T c + b^T d$$

$$\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx$$

$$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_c^b g(x)\, dx$$

$$\frac{d}{dx} c\, f(x) = c \frac{d}{dx} f(x)$$

$$\frac{d}{dx}\left(f(x) + g(x) \right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

$$\sin(A+B) = \cancel{\sin(A) + \sin(B)} \ ?$$

$$= \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\sqrt{A+B} = \sqrt{A} + \sqrt{B} \ ?$$

$$f(x) = 7x$$

$$f(a+b) = 7(a+b)$$

$$= 7a + 7b$$

$$= f(a) + f(b)$$

$$f(82a) = 7 \cdot 82a$$

$$= 82 \cdot 7 \cdot a$$

$$= 82\, f(a)$$

A function $\underbrace{f : \mathbb{R}^n \to \mathbb{R}}$

"f is a function from $\mathbb{R}^n$ to $\mathbb{R}$"

$\mathbb{R}^n$: all vectors of length n

$\mathbb{R}$: all real number

is <u>linear</u> if

* $f(x+y) = f(x) + f(y)$ ~~for~~ for all $x, y \in \mathbb{R}^n$
* $f(cx) = c f(x)$ for all $c \in \mathbb{R}$, $x \in \mathbb{R}^n$

Non-examples

$f(a,b) = a \cdot b \qquad a, b \in \mathbb{R}$

$n = 2$

$$f(1,0) = 0$$
$$f(0,1) = 0$$
$$f(1,1) = 1 \cdot 1$$
$$= 1$$

$$f(1,1) = f(1,0) + f(0,1)$$
$$0 + 0 = 0$$
$$(1,0) + (0,1) = (1,1)$$

$$f(x+y) = f(x) + f(y)$$
$$f((1,0) + (0,1)) \stackrel{?}{=} f(1,0) + f(0,1)$$
$$f(1,1) \stackrel{?}{=} f(1,0) + f(0,1)$$
$$1 \stackrel{?}{=} 0 + 0$$

No.

$$(3,-4)^T (a,b)$$

Example

$$f(a,b) = 3a - 4b$$

$$f(x) = 7x$$

$$f((a,b)) =$$

$$
\begin{aligned}
f(\alpha (a,b)) &= f((\alpha a, \alpha b)) \\
&= 3(\alpha a) - 4(\alpha b) \\
&= \alpha 3a - \alpha 4b \\
&= \alpha (3a - 4b) \\
&= \alpha f((a,b))
\end{aligned}
$$

$$f((a_1,b_1)) + f((a_2,b_2))$$ "additivity"

$$= 3a_1 - 4b_1 + 3a_2 - 4b_2$$

$$= 3(a_1+a_2) - 4(b_1+b_2)$$

$$= f((a_1+a_2, b_1+b_2))$$

$$= f((a_1,b_1)+(a_2,b_2))$$

E.g. $a \in \mathbb{R}^n$ (fixed) (like $(3,-4)$ above)

$$f(x) = a^T x \qquad (x \in \mathbb{R}^n)$$

This is a linear function!

$$f(\alpha x) = a^T(\alpha x)$$
$$= \alpha\, a^T x$$
$$= \alpha\, f(x)$$

$$f(x+y) = a^T(x+y)$$
$$= a^T x + a^T y$$
$$= f(x) + f(y)$$

(In fact, every linear map from $\mathbb{R}^n$ to $\mathbb{R}$ has this form)

Example: time series T_k temperature at time t_k

$$T = (T_1, T_2, \ldots, T_n)$$

---

map: tell me the temperature at time t_7

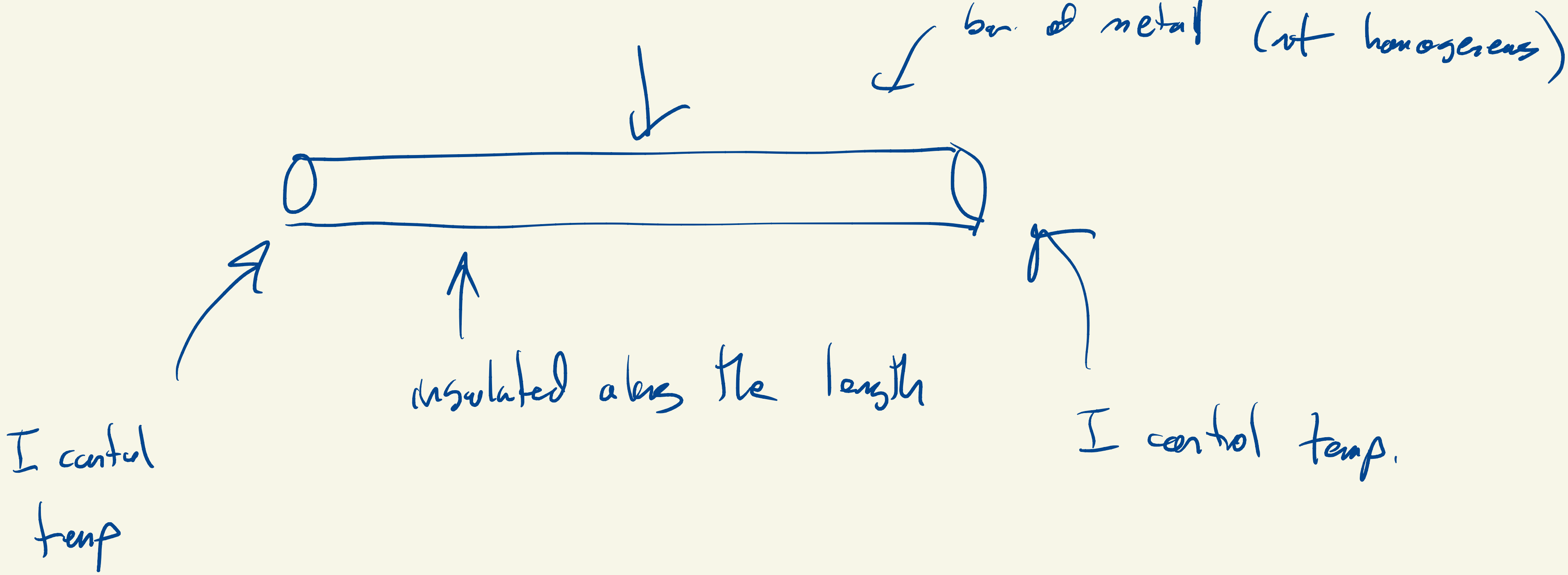
$$T \longrightarrow T_7$$

$$e_7 = (0, 0, \ldots, 0, 1, 0, \ldots 0)$$

slot 7

$$e_7^T T = 0 \cdot T_1 + 0 \cdot T_2 + \cdots + 0 \cdot T_6 + 1 \cdot T_7 + 0 \cdot T_8 + \cdots + 0 \cdot T_n$$

$$= T_7$$


bar of metal (not homogeneous)
insulated along the length
I control temp
I control temp.