1. Consider the matrix
$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 2 & 4 & 5 & 10 & 17 \end{bmatrix}$$
Reduce this matrix to echelon form. Then find all solutions of $Ax = 0$. (That is, find pivot and free variables, and then determine the special elements of the nullspace corresponding to the free variables.)

2. For the same matrix A as in the previous problem, one solution of $Ax = (1, 3, 6)$ is $x = (5, -4, 3, -2, 1)$. Find all solutions.

3. Suppose A is an $m \times n$ matrix and that W is an invertible $m \times m$ matrix. Show that the nullspace of A and the nullspace of WA are the same as each other. What can be said about the nullspace of WA instead if you don't know that W is invertible?

4. Find a 3×3 matrix F such that $Fx = (x_1, x_2 - 2x_1, x_3 + 4x_1)$. Also, compute F^{-1} using any technique that seems convenient, including the method of making an educated guess!

5. Suppose A is a 3×5 matrix. Find a 3×3 matrix E_1 such that the rows of $E_1 A$ are as follows:

1. Row 1 is row 1 of A
2. Row 2 is row 2 of A minus twice row 1 of A.
3. Row 3 is row 3 of A plus four times row 1 of A

The matrix E_1 is called an elimination matrix. You should find that E_1 is lower-triangular, and that most of the interesting things happen in the first column. Also, compute E_1^{-1}.

6. As in the problem above, find a different elimination matrix E_2 such that the first two rows of $E_2 A$ are those of A, but the third row is row three of A minus 6 times row 2 of A.

7. Ok, let's go back to the matrix A of the first problem:
$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 2 & 4 & 5 & 10 & 17 \end{bmatrix}$$
Find an elimination matrix E_1 such that $E_1 A$ is the first step of elimination so that the first column of A is cleared below the pivot. Your matrix E_1 will have ones on the diagonal and the only other non-zero entries will be in the first column.

8. Continuing with the previous problem, at this point you will have found that column 2 is already cleared. Yay! Find an elimination matrix E_2 that does the work of clearing column three from here, at which point elimination will be done. That is, $E_2 E_1 A = B$, where B is the echelon form matrix you computed in question 1.

9. From the previous problem, we know that $E_2E_1A = B$, where B is in echelon form. That means that $A = E_2^{-1}E_1^{-1}B$. Compute each of E_1^{-1}, E_2^{-1} and the product $E_2^{-1}E_1^{-1}$. The result will be a lower triangular matrix L. There will be ones on the diagonal, and the entries below the diagonal will be closely tied to numbers you saw in the process of doing elimination.

10. Suppose you want to find a solution of $Ax = (1,3,6)$. One can do this with QR factorization and the pseudo inverse. But here's another way using the factorization $A = LB$ we just formed.

- First, solve $Lw = (1,3,6)$ by using forward substitution. Do this!
- Now find a solution of $Bx = w$, for then $Ax = LBx = Lw = (1,3,6)$. To find this solution, extract the three pivot columns of B, and the result will be an upper-triangular matrix U. Great. Now use back substitution to solve $U\tilde{x} = w$. This will determine the pivot variables in $\tilde{x}$, and the solution x has its pivot variables from $\tilde{x}$ and its free variables all zero. Write down the solution x you just found.

11. Your solution to $Ax = (1,3,6)$ from the previous problem differs from mine, which was $(5,-4,3,-2,1)$. So the two solutions differ by an element of the nullspace. Show that this element of the nullspace really was in the nullspace you identified way back in Problem 1. The easiest way to do this will be to work with the "special" elements of the nullspace you will have identified back in Problem 1 associated with the free variables x_2 and x_4.