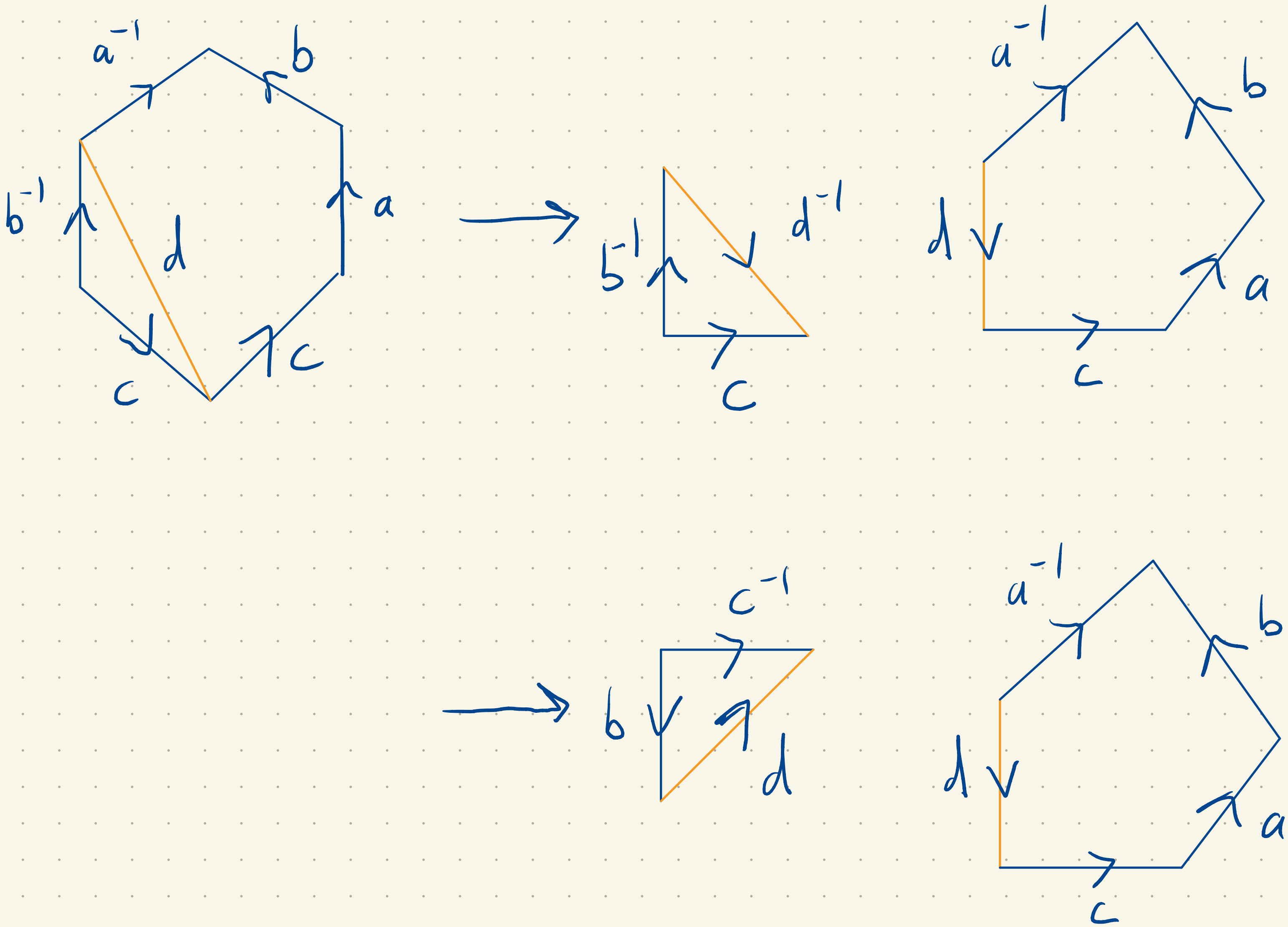
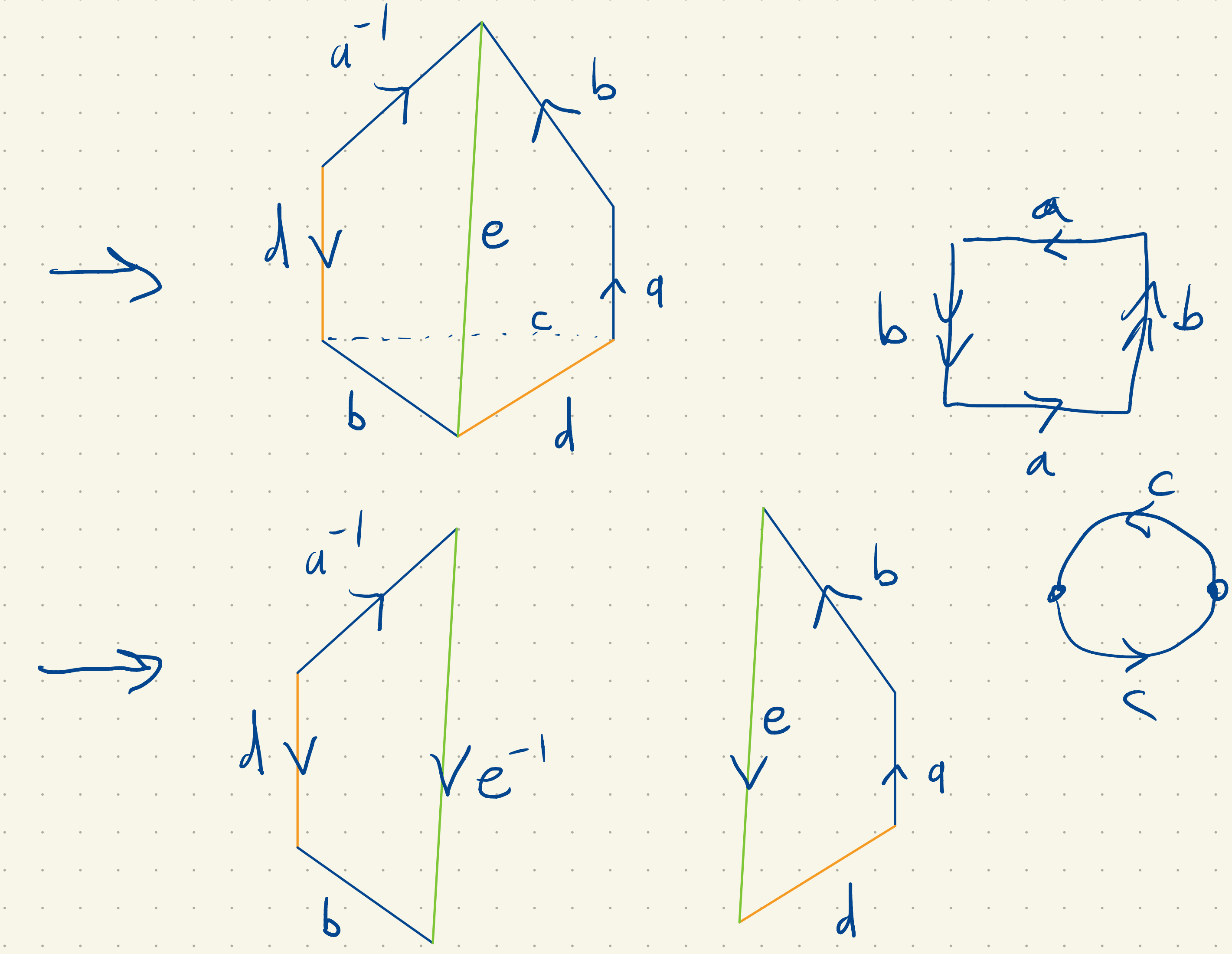
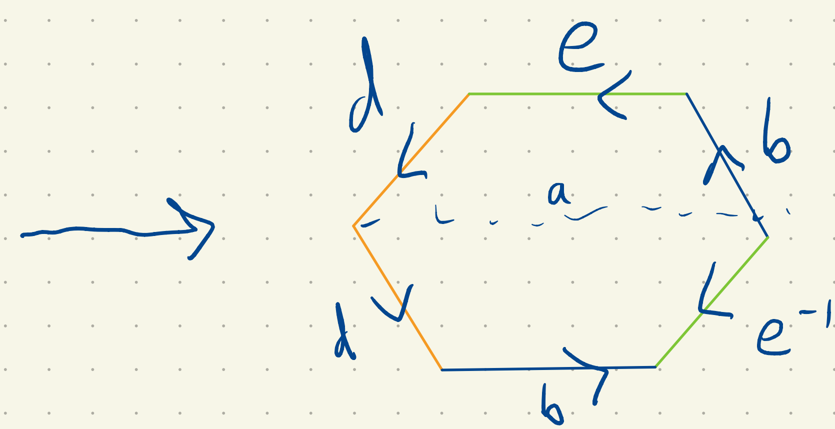
$$\mathbb{T}^2 \# \mathbb{P}^2 \sim \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$$





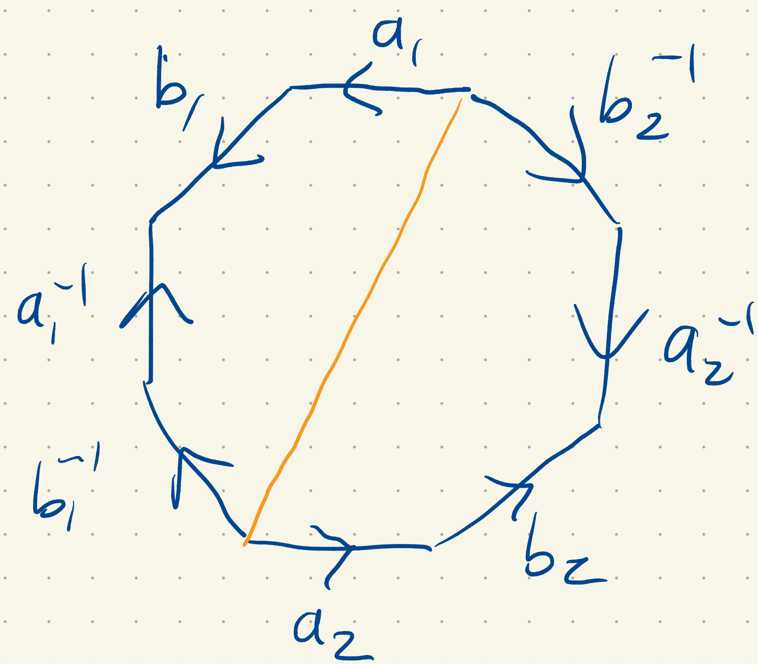


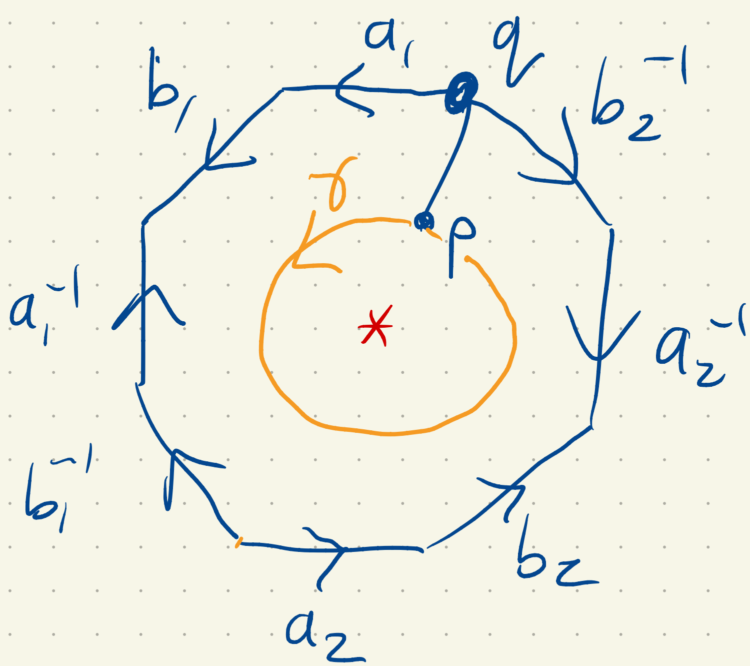
$$\sim \mathbb{P}^2 \# K$$

$$\sim \mathbb{P}^2 \# (\mathbb{P}^2 \# \mathbb{P}^2)$$

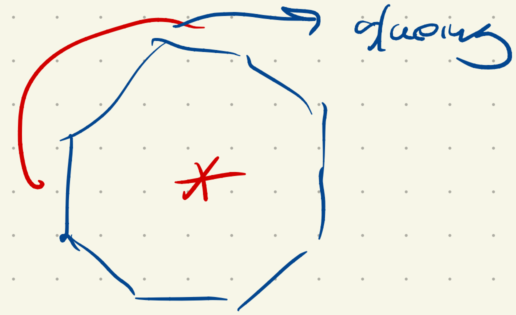
$\mathbb{T}^2 \# \mathbb{T}^2$

$\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2 \quad (\mathbb{T}^2 \# \mathbb{P}^2)$

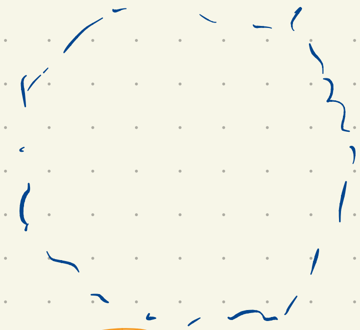




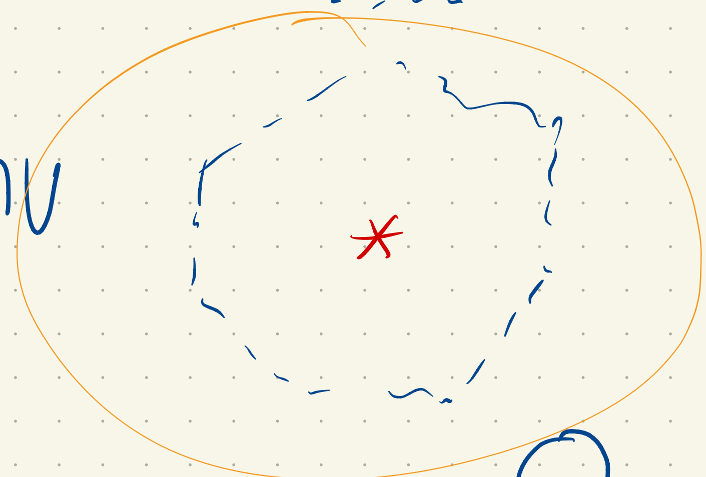
U



V

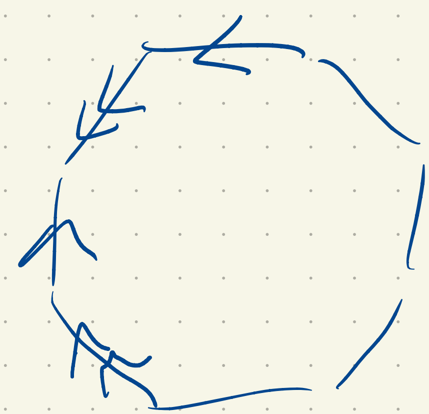


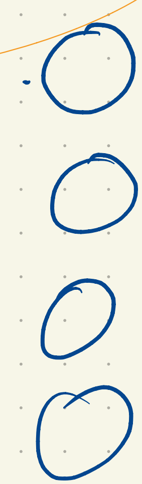
$U \cap V$



$\pi_1(V, p)$ is trivial.

$\pi_1(U, q)$





$$\pi_1(U,p) \simeq \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{4 \text{ copies}}$$

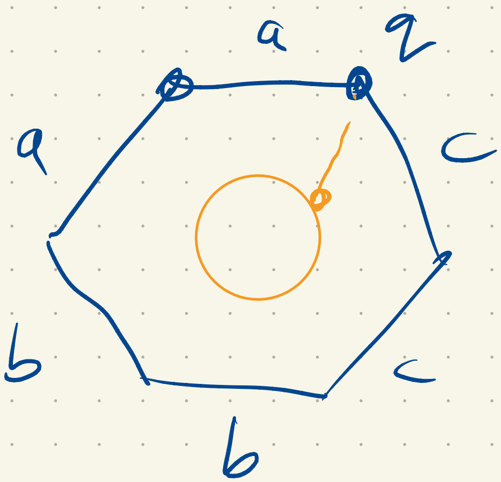
$$\pi_1(Y,p) \simeq \pi_1(U,p) / \pi_1(U \cap V, p)$$

$$\simeq \langle a_1, b_1, a_2, b_2 \mid \quad \rangle$$

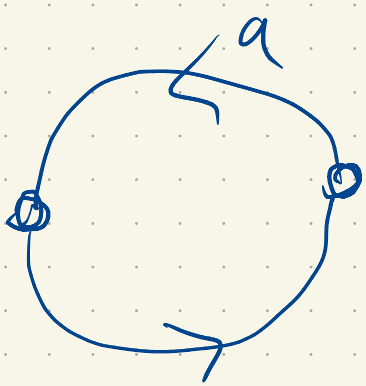
$U \cap V$ is homotopy equiv to a circle

$$\pi_1(U \cap V, p) \simeq \mathbb{Z} \qquad \langle \gamma \mid \emptyset \rangle$$

$$\pi_1(X,p) \simeq \langle a_1, b_1, a_2, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \rangle$$



$$\pi_1(X,p) \simeq \langle a, b, c \mid a^2 b^2 c^2 \rangle$$



$$\langle a \mid a^2 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$$

Let G be a group.

It commutator subgroup $[G,G] = \overline{\{ghg^{-1}h^{-1} : g,h \in G\}}$

We can form

$$Ab(G) = G/[G,G]$$

$ghg^{-1}h^{-1} \sim 1$

$gh \sim hg$

Claim: $Ab(G)$ is abelian.

$$\begin{aligned} g[G,G]\,h[G,G] &= gh[G,G] \\ &= gh\,h^{-1}g^{-1}hg\,[G,G] \\ &= hg\,[G,G] \\ &= h[G,G]\,g[G,G] \end{aligned}$$

Characteristic Property: (of Abelianization)

Let G be a group and H be an abelian group. Given a hom $\tilde{\phi}: G \to H$ there is a unique homomorphism $\phi: Ab(G) \to H$ s.t.

$$\begin{tikzcd} G \arrow[d] \arrow[dr, "\tilde{\phi}"] & \\ Ab(G) \arrow[r, "\phi"'] & H \end{tikzcd}$$

$$Ab(G) = G/[G,G]$$

Just need to show $\tilde{\phi}(g h g^{-1} h^{-1}) = 1_H$.

$$\Downarrow$$

$$\tilde{\phi}(g)\tilde{\phi}(h)\tilde{\phi}(g)^{-1}\tilde{\phi}(h)^{-1}$$

$$\tilde{\phi}(g)\tilde{\phi}(g)^{-1}\tilde{\phi}(h)\tilde{\phi}(h)^{-1} = 1_H$$

We'll show $Ab(\langle a_1, b_1, a_2, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \rangle) \sim \mathbb{Z}^4$

$Ab(\langle a, b, c \mid a^2 b^2 c^2 \rangle) \sim \mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})$

$$\underbrace{Ab\left(\langle a_1, b_1, a_2, b_2 \mid \underbrace{[a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}]}_{\to R} \rangle\right)}_{G} \simeq \mathbb{Z}^4$$

$$F(\{a_1, b_1, a_2, b_2\}) \xrightarrow{\tilde{\phi}} \mathbb{Z}^4$$

$$\begin{aligned}
a_1 &\to e_1 = (1,0,0,0) \\
a_2 &\to e_2 = (0,1,0,0) \\
b_1 &\to e_3 \\
b_2 &\to e_4
\end{aligned}$$

$$G \xrightarrow{\tilde{\phi}} \mathbb{Z}^4$$

$$\tilde{\phi}(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}) = e_1 + e_2 - e_1 - e_2 + e_3 + e_4 - e_3 - e_4$$

$= 0$

so $\tilde{\tilde{\phi}}$ descends to $\tilde{\phi}$.

$$\begin{tikzcd} G \arrow[d] \arrow[dr, bend left, "\tilde{\phi}"] & \\ G/[G,G] \arrow[r, "\phi"'] & \mathbb{Z}^4 \end{tikzcd}$$

$$\psi : \mathbb{Z}^4 \to \mathrm{Ab}(G)$$

$$e_1 \to [a_1] = (a_1 \overline{R})[G,G]$$

$$e_2 \to [a_2]$$

$$e_3 \to [b_1]$$

$$e_4 \to [b_2]$$

$$c_1 e_1 + c_2 e_2 \cdots + c_4 e_4 \to [a_1]^{c_1} \cdots [b_2]^{c_4}$$

$$\varphi(\psi(e_1)) = \varphi(\underline{[a_1]}) = \tilde{\varphi}(a_1) = e_1$$

$$\begin{tikzcd}
F(\{a_1, b_1, a_2, b_2\}) \arrow[d] \arrow[rrdd, "\tilde{\tilde{\phi}}"] & & \\
G \arrow[d] \arrow[rrd, "\tilde{\phi}"] & & \\
Ab(G) \arrow[rr, "\phi"'] & & \mathbb{Z}^4
\end{tikzcd}$$

Now rinse + repeat

$$\varphi \circ \psi(e_i) = e_i \qquad i = 1, \dots, 4.$$

Exercise: $\varphi \circ \psi$ is the identity.

$$\psi(\varphi([a_1])) = \psi(\tilde{\varphi}(a_1))$$
$$= \psi(e_1)$$
$$= [a_1]$$

ditto: $[a_2] \to [a_2]$

$[b_1] \to [b_1]$

$[b_2] \to [b_2]$

Exercise: $\psi \circ \varphi = id.$

ψ, φ are inverses.

$$\mathrm{Ab}(\underbrace{\langle a,b,c \mid a^2b^2c^2\rangle}_{G})$$

$$\varphi : \mathrm{Ab}\, G \to \mathbb{Z}^2 \oplus \mathbb{Z}_2$$

$f+f=0$

$$\tilde{\varphi}(a) = e_1$$

$$\tilde{\varphi}(b) = e_2$$

$$\tilde{\varphi}(c) = f - e_1 - e_2$$

$$\tilde{\varphi}(a^2b^2c^2) = e_1 + e_1 + e_2 + e_2 + f - e_1 - e_2 + f - e_1 - e_2$$

$$= 2e_1 - 2e_1 + 2e_2 - 2e_2 + 2f$$

$$= 0$$

$$Ab(G) \xrightarrow{\varphi} \mathbb{Z}^2 \oplus \mathbb{Z}_2$$

$$\mathbb{Z}^2 \oplus \mathbb{Z}_2 \to Ab(G)$$

$$e_1 \to [a] \leftarrow (a\bar{R})[G,G]$$

$$e_2 \to [b]$$

$$f \to [abc]$$

This extends to a hom. ψ.

$$\varphi(\psi(f)) = \varphi([abc])$$
$$= \tilde{\varphi}(abc)$$

$$= e_1 + e_2 + f - e_1 - e_2$$
$$= f$$

$$\psi(\varphi([c])) = \psi(\tilde{\varphi}(c)) = \psi(f - e_1 - e_2)$$
$$= [abc][a]^{-1}[b]^{-1}$$
$$= [c]$$

$\psi(\varphi([a])) = [a]$ and also for $[b]$

$\varphi(\psi(e_i)) = e_i \quad i = 1, 2.$

Exercise $\psi \circ \varphi = id$

$\varphi \circ \psi = id.$

$$G \sim \mathbb{Z}^2 \oplus \mathbb{Z}_2$$

$\underbrace{T^2 \# \cdots \# T^2}_{n}$ $\quad \mathbb{Z}^{2n}$

$\underbrace{\mathbb{P}^2 \# \cdots \# \mathbb{P}^2}_{n}$ $\quad \mathbb{Z}^{n-1} \oplus \mathbb{Z}_2$