$$\langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle \simeq \langle S_1 \mid R_1 \rangle * \langle S_2 \mid R_2 \rangle$$

$$F(S_1 \cup S_2) / \overline{R_1 \cup R_2} \qquad\qquad F(S_1)/\overline{R_1}$$

$$s_i \in S_1 \qquad\qquad s_i \bmod \overline{R_1}$$

Free product $\boxed{\underset{\alpha \in I}{\ast} G_\alpha}$

Free group (on certain generators)

$$F(S)$$
$$\uparrow$$
"stuff"

$$F(\sigma) = \{ (\sigma, n) : n \in \mathbb{Z} \}$$

$$\sigma^n \longleftrightarrow (\sigma, n)$$

$$\sigma^n \cdot \sigma^m = \sigma^{n+m} = (\sigma, n+m)$$

$$\underset{\sigma \in S}{\ast} F(\sigma)$$

$$F(\sigma) \to \mathbb{Z}$$

$$\phi(\sigma^n) = n$$

$$\sigma^n = 1 \iff n = 0$$

$F(S)$ $\qquad |S| \geq 2$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

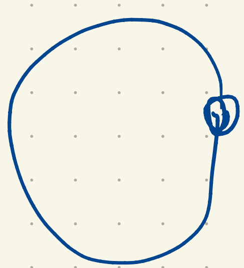
$$S \sim S'$$

$$F(S) \sim F(S')$$

$$\begin{array}{ccc} G_1 & & G_2 \\ f_1 \downarrow & & \downarrow f_2 \\ H_1 & & H_2 \end{array}$$

$$
\begin{tikzcd}
G_1 * G_2 \arrow[r] & H_1 * H_2 \\
G_i \arrow[u, hook] \arrow[r, "f_i"] & H_i \arrow[u, hook]
\end{tikzcd}
$$

$$\overbrace{\sigma_1^{n_1}\sigma_2^{n_2}\cdots\sigma_j^{n_j}}^{1}$$



$\sigma_i \neq \sigma_{i+1}$

$n_i \neq 0 \quad \forall i$

$$\begin{tikzcd} F(S) \arrow[dr, bend left, "\tilde{\phi}"] & \\ \langle S \mid R \rangle \arrow[r] & H \end{tikzcd}$$

$$\pi_1(U, p) \cong \langle S_1 \mid R_1 \rangle$$

$$\pi_1(V, p) \cong \langle S_2 \mid R_2 \rangle$$

$$\pi_1(U \cap V, p) \cong \langle S_3 \mid R_3 \rangle$$

Claim $\pi(X, p) \cong \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup R' \rangle$

To describe R':

Take $s \in S_3$.

$$s \to [\gamma_s]_{U \cap V} \xrightarrow{i_*} [\gamma_s]_U \longrightarrow \alpha_s \overline{R_1}$$

$$\alpha_s \in F(S_1)$$

$$g \to [\gamma_s]_{U\cap V} \xrightarrow{j_*} [\gamma_s]_V \to \beta_s \overline{R_2}$$

$$\beta_s \in F(S_2)$$

$$R' = \{ \alpha_s \beta_s^{-1} : s \in S_3 \}$$

$K \trianglelefteq G$

$$\begin{tikzcd}
G \arrow[d, "\pi"'] \arrow[dr, "\tilde{\phi}"] & \\
\underline{G/K} \arrow[r, dashed, "\phi"'] & H
\end{tikzcd}$$

$k \in K$

$$\tilde{\phi}(k) = \{1_H\}$$

$$\ker \tilde{\phi} \supseteq K$$

$$\phi(\pi(g)) := \tilde{\phi}(g)$$

$$\pi(g_1) = \pi(g_2) \Rightarrow g_1 = g_2 k$$

$$\tilde{\phi}(g_1) = \tilde{\phi}(g_2 k)$$
$$= \tilde{\phi}(g_2)\underbrace{\tilde{\phi}(k)}_{1}$$

Special cases:

1) If $U \cap V$ is simply connected

$$\pi_1(X,p) \cong \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$$

2) If V is simply connected

$$\pi_1(X,p) \cong \langle S_1 \mid R_1 \cup R' \rangle$$

$$R' = \{ \alpha_s : s \in S_3 \}$$ as above

$$
\begin{tikzcd}
 & \pi_1(U,p) \arrow[d] \arrow[dr, bend left] & \\
\pi_1(U\cap V,p) \arrow[ur, "i_*"] \arrow[dr, "j_*"'] & \pi_1(U,p) * \pi_1(V,p) \arrow[r, "\Phi"] & \pi_1(X,p) \\
 & \pi_1(V,p) \arrow[u] \arrow[ur, bend right] &
\end{tikzcd}
$$

$$\ker \Phi = \overline{C}$$

$$C = \{\, i_* \gamma (j_* \gamma)^{-1} : \gamma \in \pi_1(U\cap V, p) \,\}$$

Φ is surjective

$$\ker \Phi = \overline{C}$$

"amalgamated free product"

$$\begin{tikzcd} & G_1 \arrow[d] \\ H \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & G_1 * G_2 \\ & G_2 \arrow[u] \end{tikzcd}$$

$$G_1 *_H G_2$$

$$\hookrightarrow \quad G_1 * G_2 \big/ \overline{\{ \varphi_1(h) \varphi_2(h)^{-1} : h \in H \}}$$

Claim

$$G_1 = \langle S_1 \mid R_1 \rangle$$

$$G_2 = \langle S_2 \mid R_2 \rangle$$

$$H = \langle S_3 \mid R_3 \rangle$$

then $\quad G_1 *_H G_2 \simeq \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup R' \rangle$

$s \in S_3$. $\quad \varphi_1(s) = \alpha_s \overline{R_1}$ $\quad R' = \{ \alpha_s \beta_s^{-1} : s \in S_3 \}$

$\varphi_2(s) = \beta_s \overline{R_2}$

Lemma: $\mathcal{C}' = \{ \varphi_1(s)\varphi_2(s)^{-1} : s \in S_3 \}$

$\mathcal{C} = \{ \varphi_1(h)\varphi_2(h)^{-1} : h \in H \}$

Claim: $\overline{\mathcal{C}} = \overline{\mathcal{C}'}$

Key step If $s \in S_3$ then $\varphi_1(s^{-1})\varphi_2(s^{-1})^{-1} \in \overline{\mathcal{C}'}$.

$$\varphi_1(s)^{-1}\varphi_1(s)\varphi_2(s)^{-1}\varphi_1(s) \in \overline{\mathcal{C}'}$$

$$\varphi_2(s)^{-1}\varphi_1(s) \in \overline{\mathcal{C}'}$$

$$\left(\varphi_1(s)^{-1}\varphi_2(s)\right)^{-1} \in \overline{\mathcal{C}'}$$

$$\varphi_1(s)^{-1}\varphi_2(s) \in \overline{\mathcal{C}'}$$

$$\|$$

$$\varphi_1(s^{-1})\varphi_2(s^{-1})^{-1}$$

Remainder: up to you!

Exercise 9.4 b)

$$\langle S_1 \mid R_1 \rangle * \langle S_2 \mid R_2 \rangle \cong \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$$

$$s \in S_1$$

$$S_1 \; \overline{R_1} \longrightarrow S_1 \; \overline{R_1 \cup R_2}$$

Exercise 9.5

$$S, \quad R \subseteq F(S) \qquad \pi : F(S) \to \langle S \mid R \rangle$$

$$R' \subseteq F(S)$$

$$\langle S \mid R \rangle \big/ \overline{\pi(R')} \cong \langle S \mid R \cup R' \rangle$$

goal: $G_1 *_H G_2 \cong \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup R' \rangle$

$$G_1 *_H G_2 = \langle S_1 | R_1 \rangle * \langle S_2 | R_2 \rangle / \overline{C}$$

$$= \underbrace{\langle S_1 | R_1 \rangle * \langle S_2 | R_2 \rangle / \overline{C'}}_{A}$$

Now $\langle S_1 | R_1 \rangle * \langle S_2 | R_2 \rangle \cong \langle S_1 \cup S_2 | R_1 \cup R_2 \rangle$

$$C' = \{ (\alpha_s \overline{R_1})(\beta_s \overline{R_2})^{-1} : s \in S_3 \}$$

$$\downarrow$$

$$C'' = \{ (\alpha_s \overline{R_1 \cup R_2})(\beta_s \overline{R_1 \cup R_2})^{-1} : s \in S_3 \}$$

$$(A) \cong \underbrace{\langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle / \overline{C''}} \longrightarrow (B)$$

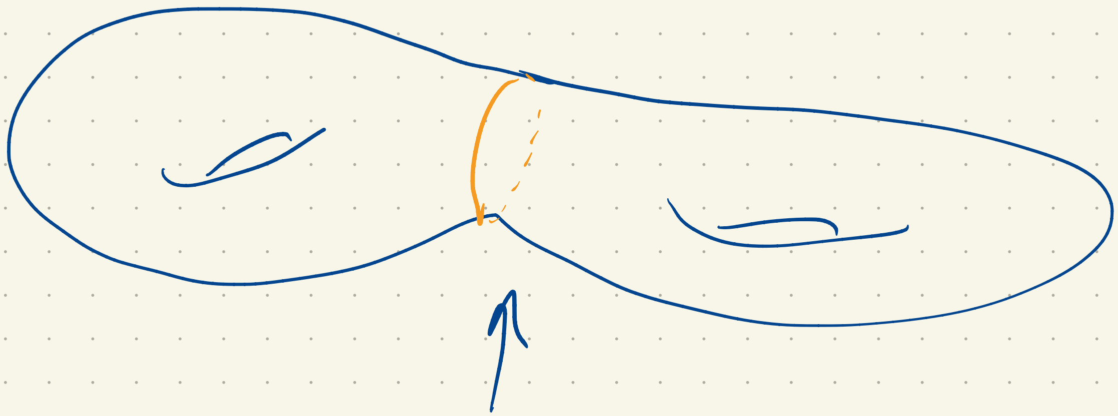
$$C'' = \pi\left(\{\alpha_s \beta_s^{-1} : s \in S_3\}\right).$$

$$\pi : F(S_1 \cup S_2) \to \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle$$

$$(B) \cong \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup R' \rangle$$

$$R' = \{\alpha_s \beta_s^{-1} : s \in S_3\}.$$

New construction



$M_1 \# M_2$ connected sum,

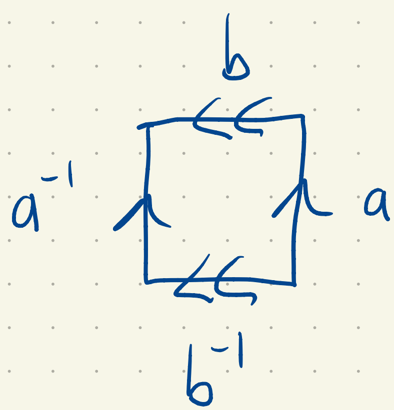
It is a manifold.

Every compact 2-manifold is homeomorphic to one of

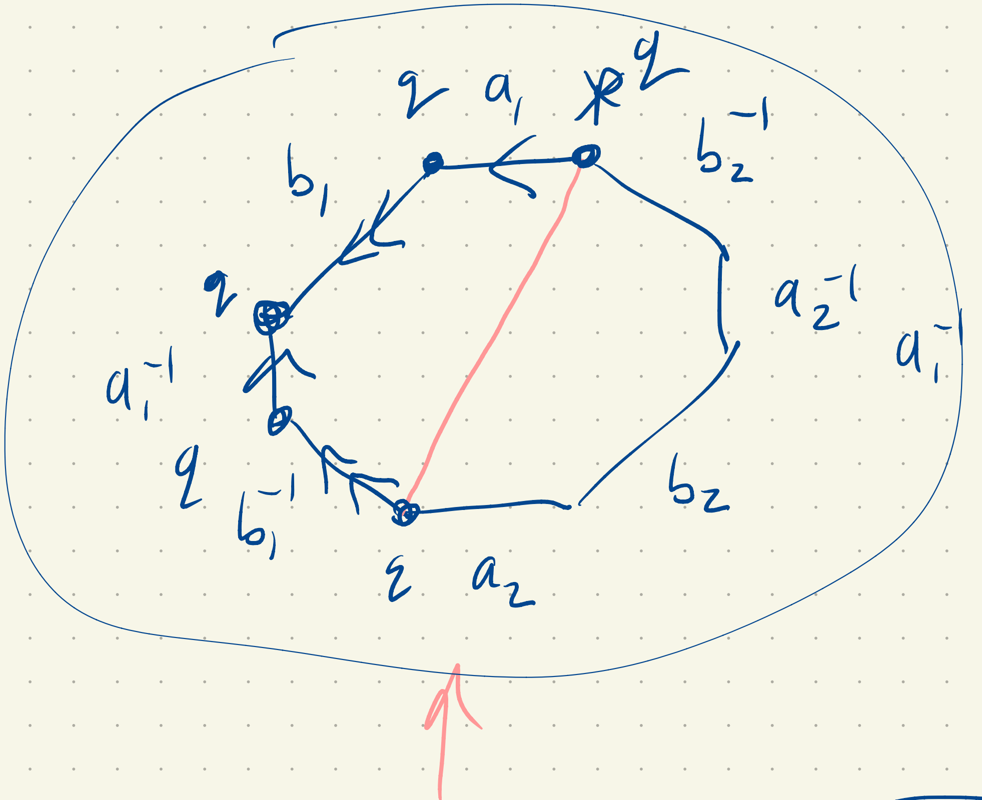
a) S^2

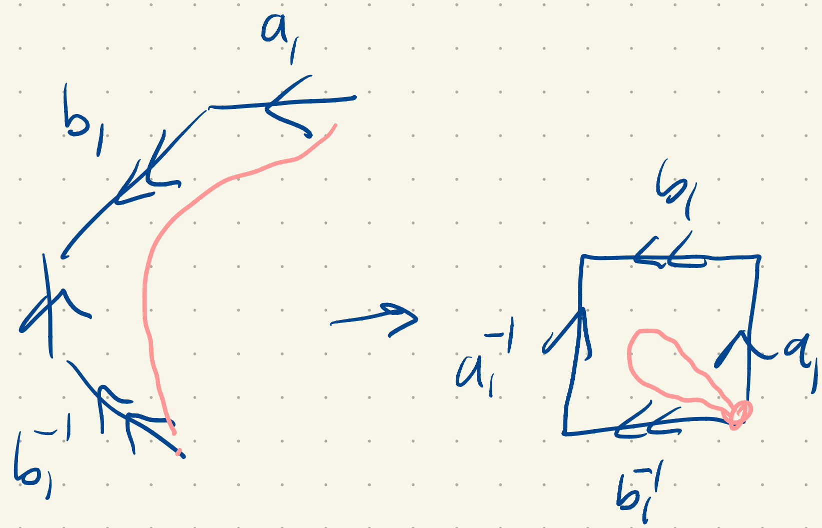
b) $\mathbb{T}^2 \# \cdots \# \mathbb{T}^2$

c) $\mathbb{P}^2 \# \cdots \# \mathbb{P}^2$

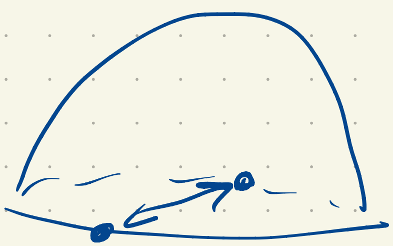


$$\longleftrightarrow \langle a, b \mid a b a^{-1} b^{-1} \rangle$$

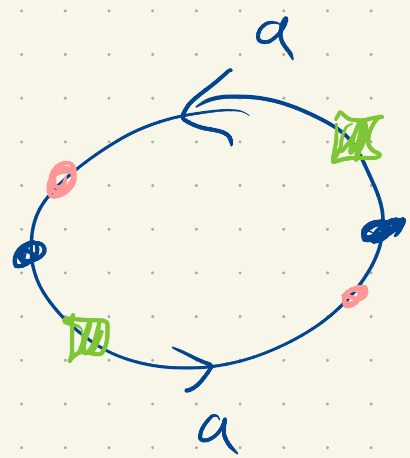


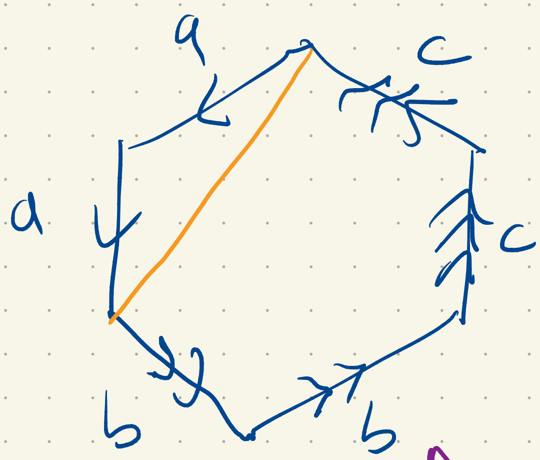


$\mathbb{T}^2 \# \mathbb{T}^2$

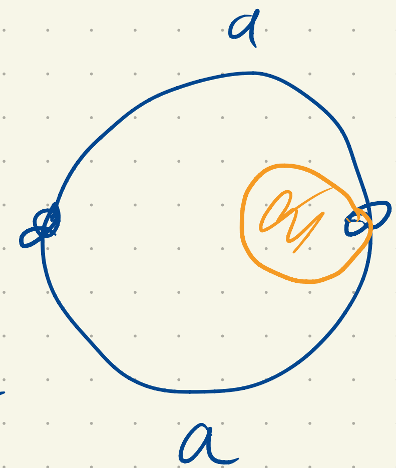


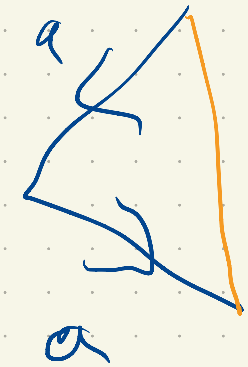
$\sim \quad \mathbb{P}^2$



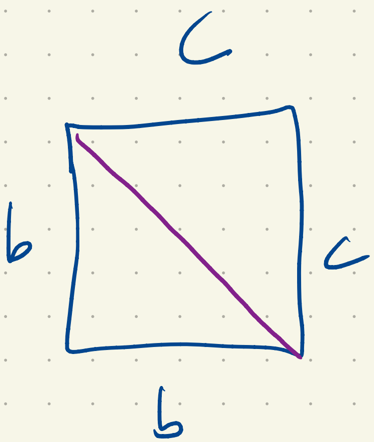


$\langle a, b, c \mid a^2 b^2 c^2 \rangle$

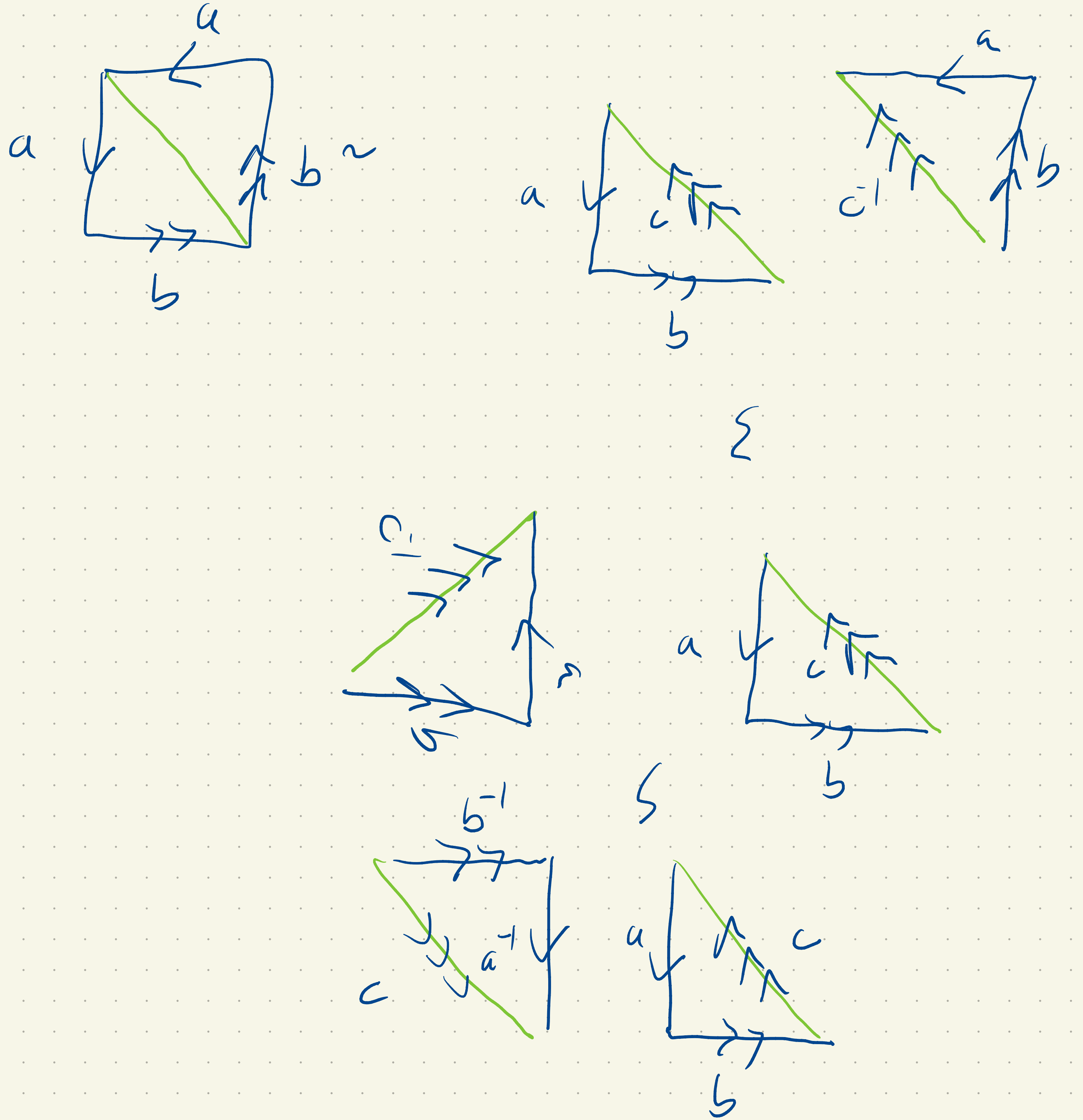




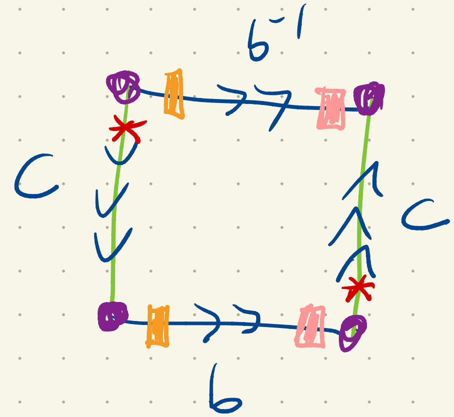
$\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$

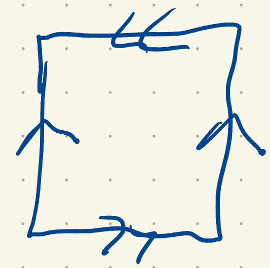


$\mathbb{P}^2 \# \mathbb{P}^2$



a
a
b
b
~
a
c
b
a
c⁻¹
b
c⁻¹
a
b
a
c
b
b⁻¹
a⁻¹
c
a
c
b





$$\mathbb{P}^2 \# \mathbb{T}^2 \longrightarrow \mathbb{P}^2 \# K$$

$$\cong$$

$$\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$$