$$\mathcal{C} = \{ [\gamma]_U [\gamma]_V^{-1} : \gamma \text{ is a loop in } U \cap V \text{ with base point } p \}$$

$$\ker \Phi \supseteq \mathcal{C} \Rightarrow \ker \Phi \supseteq \overline{\mathcal{C}}$$

Siefert Van Kampen Theorem:

Φ is surjective and $\ker \Phi = \overline{\mathcal{C}}$

so $\pi_1(X, p) \cong \pi_1(U,p) * \pi_1(V,p) / \overline{\mathcal{C}}$.

R' consists of $s_1 s_2^{-1}$

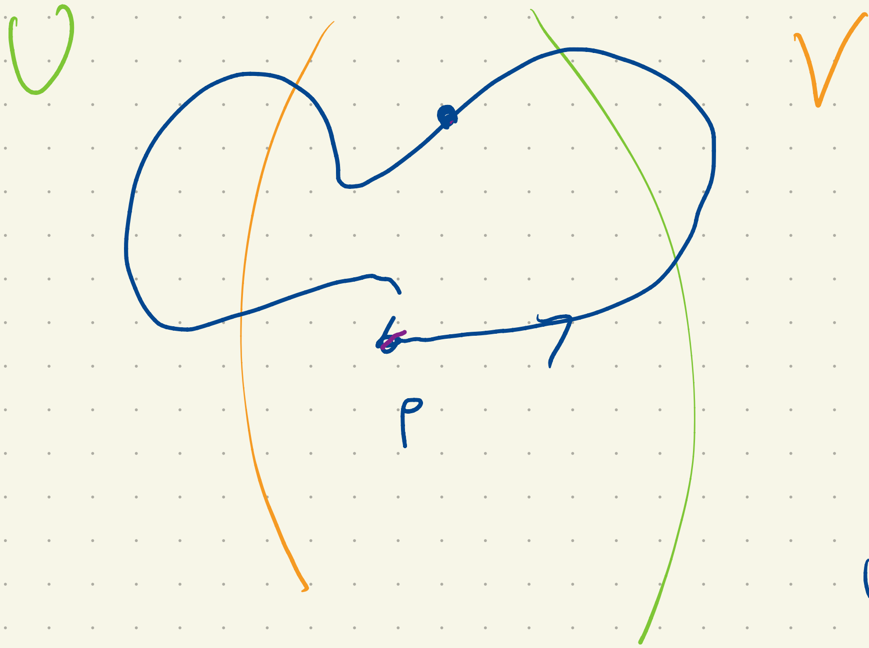
Surjectivity of Φ

Given $\alpha \in \pi_1(X, p)$ want to show

$$\alpha = \Phi\left([\alpha_1]_U \cdot \, \cdots \, \overset{*}{\cdot} [\alpha_n]_V \right)$$

$$= [\alpha_1]_X \cdot \qquad \cdot [\alpha_n]_X$$

$\alpha = [a] \qquad a \longrightarrow$ loop in X based at p

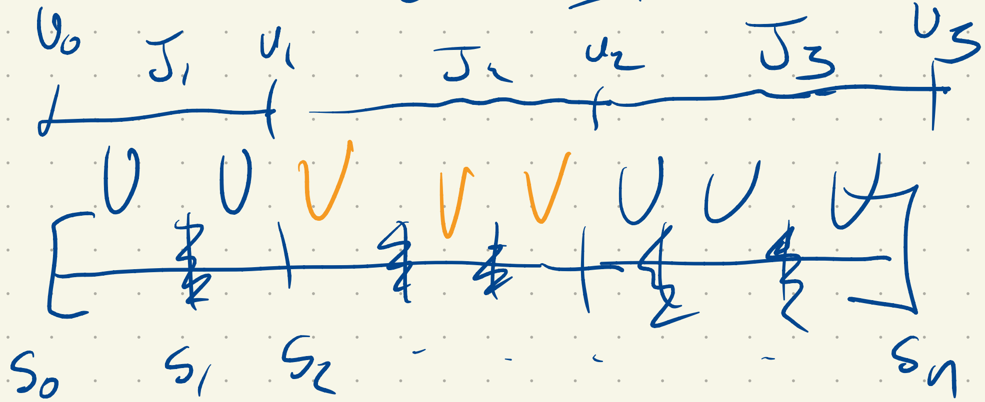


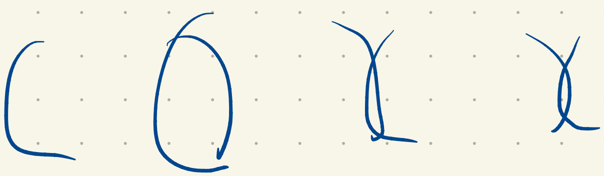
$a: I \to X$

U, V

$\underbrace{a^{-1}(U) \quad a^{-1}(V)}$

↳ cover I





$$a([s_{i-1}, s_i]) \subseteq U \text{ or } V$$

$$[s_{i-1}, s_i] \subseteq a^{-1}(U) \text{ or } a^{-1}(V)$$

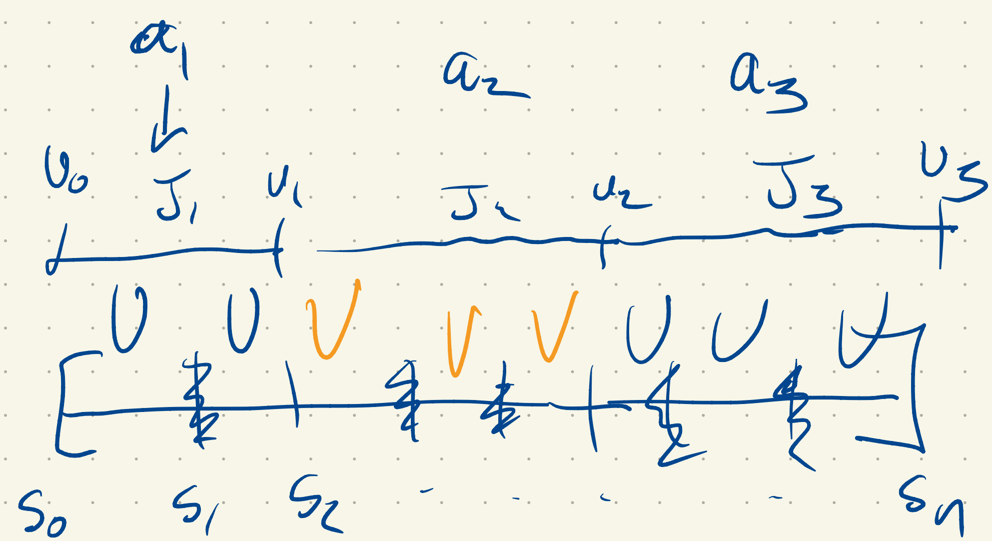
Now consolodate, $\underbrace{a(u_i)}_{p_i} \in U \cap V$ for all i.

$p_0 = p \qquad p_i \in U \cap V \qquad p_k = p$

For each p_i find a path c_i from p to p_i

($c_i = c_p$ if $p_i = p$)

Let



Let $\hat{a}_i = c_{i-1} \cdot a_i \cdot \bar{c}_i$

$$[a]_X = [a_1]_X \cdot \dots \cdot [a_n]_X \qquad \underline{\bar{c}_i \quad c_i}$$

$$= \overbrace{[c_p]_X [a_1]_X [c_1]^{-1}}^{[\hat{a}_1]} \, \overbrace{[c_1] [a_2] [c_2]^{-1}}^{[\hat{a}_2]}$$

$$= [\hat{a}_1]_X \cdot \dots \cdot [\hat{a}_n]_X$$

Each $\hat{a}_i$ is a loop either in U or in V.

$$= \Phi([\hat{a}_1]_{W_1}) \cdots \Phi([\hat{a}_n]_{W_n})$$

$$= \Phi([\hat{a}_1]_{W_1} \cdot \dots \cdot [\hat{a}_n]_{W_n})$$

Free groups

$\sigma \leftarrow$ some object.

$F(\sigma)$ "=" $\{\sigma^n\}$ $\qquad \sigma^n \sigma^m = \sigma^{n+m}$

$S = \{a, b\, c\}$

If S is a set

$$F(S) = \ast_{\sigma \in S} F(\sigma) \qquad \sigma_1^{n_1} \sigma_2^{n_2} \sigma_3^{n_3} \dots \sigma_j^{n_j}$$

$$\sigma_i \neq \sigma_{i+1}$$

To define a homomorphism $F(\sigma) \to H$

it is enough to declare $\phi(\sigma)$

To define $F(S) \to H$

it is enough to declare $\phi(\sigma)$ $\forall \sigma \in S$

Given $R \subseteq F(S)$ we can make

$$F(S)/\overline{R} =: \langle S \mid R \rangle$$

generators relations

$ab = ba$

$$\langle a, b \mid aba^{-1}b^{-1} \rangle \simeq \mathbb{Z}^2$$

$$\langle a \mid a^2 \rangle \simeq \mathbb{Z}/2\mathbb{Z} \quad ab = ba$$

$$\mathbb{Z}_2$$

To define a homomorphism

$$\phi : \langle S \mid R \rangle \longrightarrow H$$

define $\tilde{\phi} : S \longrightarrow H$

which then gives me $\tilde{\phi} : F(S) \longrightarrow H$

Now ensure that $\tilde{\phi}(r) = id$ for all $r \in R$.

$\hookrightarrow F(S)$

Then $\ker \tilde{\phi} \supseteq \overline{R}$

$$\begin{tikzcd}
F(S) \arrow[d] \arrow[dr, "\tilde{\phi}"] & \\
F(S)/\overline{R} \arrow[r, "\phi"'] & H
\end{tikzcd}$$

Given any $\tilde{\phi} : S \to H$ there is at most one ϕ that makes the above diagram commute and it exists iff $\tilde{\phi}(R) = \{id_H\}$

Suppose

$$\pi_1(U, p) \simeq \langle S_1 \mid R_1 \rangle$$

$$\pi_1(V, p) \simeq \langle S_2 \mid R_2 \rangle$$

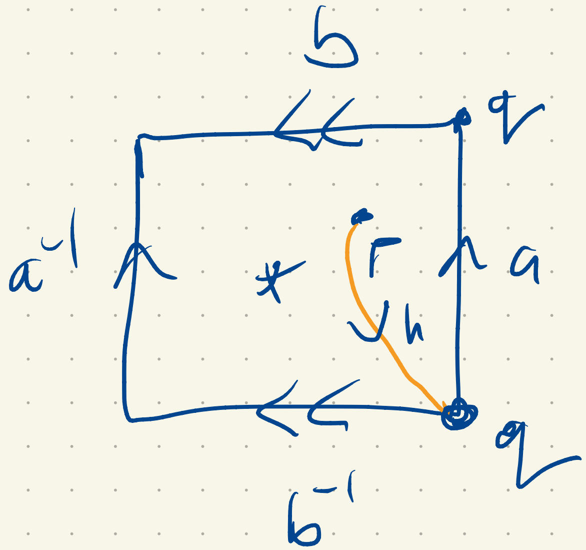
$$\pi_1(U \cap V, p) \simeq \langle S_3 \mid R_3 \rangle$$

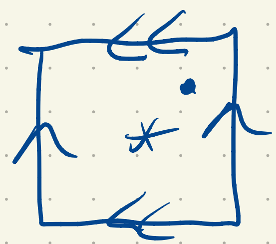
Claim: $\pi_1(X, p) \simeq \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup R' \rangle$

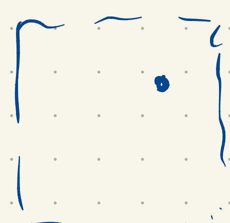
where

R' consists of $\Big\{ \underbrace{(i_* s)}_{F(S_1)} \underbrace{(j_* s)^{-1}}_{F(S_2)} : s \in S^3 \Big\} \subseteq F(S_1 \cup S_2)$

$$i_*(s) = j_*(s)$$



$U =$ 

$V =$ 

$U \cap V$ 

$\underline{\pi_1(U,p)} = \langle \hat{a}, \hat{b} \mid \rangle$

$\hat{a} = hah^{-1}$
$\hat{b} = hbh^{-1}$

$\pi_1(V,p)$ is trivial.

$\pi_1(U \cap V, p)$ is $\mathbb{Z}$ $\qquad \gamma = \hat{a}\hat{b}\hat{a}^{-1}\hat{b}^{-1}$

$\pi_1(X, p) \sim \langle \hat{a}, \hat{b} \mid R' \rangle$

$\langle \hat{a}, \hat{b} \mid \hat{a}\hat{b}\hat{a}^{-1}\hat{b}^{-1} \rangle = \mathbb{Z}^2$