$\{N_\alpha\}_{\alpha \in I}$ $\quad N_\alpha \subseteq G$, normal subgroup

Is $\bigcap_{\alpha \in I} N_\alpha$ normal? Yep!

n $\quad g^{-1} n g \in N_\alpha$ since $n \in N_\alpha$, $\forall \alpha$.

Given $C \subseteq G$, a set, maybe not even a subgroup

we can construct the smallest normal subgroup contains C.

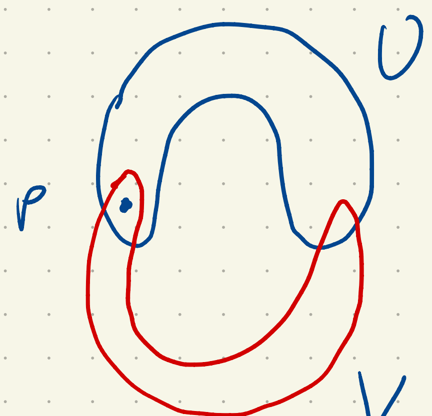
It is the intersection of all normal subgroups contains C.

It's called the normal closure of C, $\overline{C}$.

X, open sets U, V $\quad U \cup V = X$
path connected.

$p \in U \cap V$ $\quad$ (X is path connected)

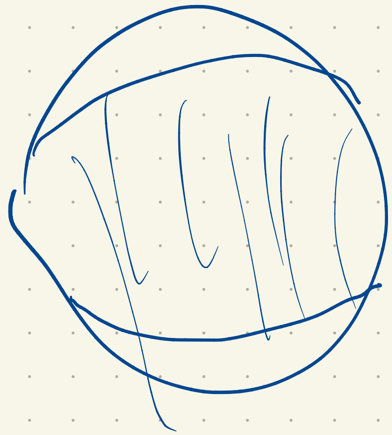
Goal: describe $\pi_1(X, p)$ in terms of $\pi_1(U, p)$ and $\pi_1(V, p)$



$\pi_1(U \cup V, p) \cong \mathbb{Z}$

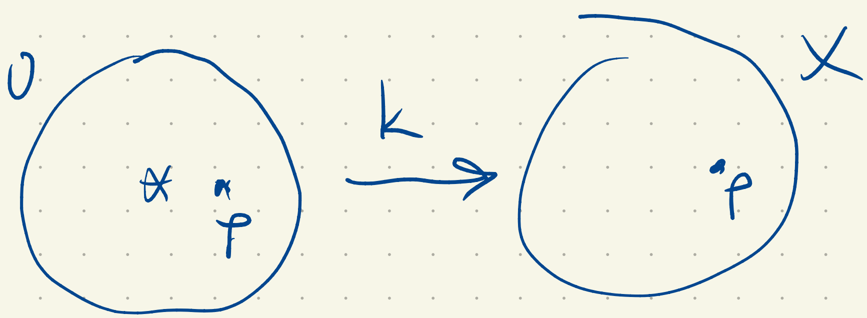
$\pi_1(U, p)$ is trivial since U is homeo to a disk.

$\pi_1(V, p)$ is trivial



Key hypothesis: $U \cap V$ is path connected

$$\begin{tikzcd}
 & U \arrow[dr, "k"] & \\
U \cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & X \\
 & V \arrow[ur, "l"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \pi_1(U,p) \arrow[dr, "k_*"] & \\
\pi_1(U \cap V, p) \arrow[ur, "i_*"] \arrow[dr, "j_*"'] & & \pi_1(X,p) \\
 & \pi_1(V,p) \arrow[ur, "l_*"'] &
\end{tikzcd}$$



$$
\begin{tikzcd}
\pi_1(U,p) \arrow[d] \arrow[dr, "k_*"] & \\
\pi_1(U,p) * \pi_1(V,p) \arrow[r, "\Phi"] & \pi_1(X,p) \\
\pi_1(V,p) \arrow[u] \arrow[ur, "l_*"'] &
\end{tikzcd}
$$

$$[\gamma_1]_U [\gamma_2]_V \cdots [\gamma_n]_V \longrightarrow [\gamma_1]_X \cdot \cdots [\gamma_n]_X$$

We will show that Φ is surjective,

From the first iso theorem

$$\pi_1(X,p) \sim \pi_1(U,p) * \pi_1(V,p) / \ker \Phi$$

$$\begin{tikzcd}
G \arrow[d] \arrow[dr, "\tilde{\phi}"] & \\
G/\ker\phi \arrow[r, "\phi"'] & H
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_1(U,p) * \pi_1(V,p) \arrow[d] \arrow[dr] & \\
\pi_1(U,p) * \pi_1(V,p)/\ker\phi \arrow[r] \arrow[r, bend right=60, leftrightarrow, "\text{iso.}"'] & \pi_1(X,p)
\end{tikzcd}$$

$$
\begin{tikzcd}
 & \pi_1(U,p) \arrow[d] \arrow[dr, bend left, "k_*"] & \\
\pi_1(U \cap V, p) \arrow[ur, "i_*"] \arrow[dr, "j_*"'] & \pi_1(U,p) * \pi_1(V,p) \arrow[r, "\Phi"] & \pi_1(X,p) \\
 & \pi_1(V,p) \arrow[u] \arrow[ur, "l_*"'] &
\end{tikzcd}
$$

$[\gamma]_{U \cap V}$

$$i_* [\gamma]_{U \cap V} = [\gamma]_U$$

$$j_* [\bar{\gamma}]_{U \cap V} = [\bar{\gamma}]_V$$

$$j_* [\gamma]_{U \cap V}^{-1} = [\gamma]_V^{-1}$$

$$[\gamma]_U [\gamma]_V^{-1} \in \pi_1(U,p) * \pi_1(V,p)$$

$$\Phi\left([\gamma]_U [\gamma]_V^{-1}\right) = \Phi([\gamma]_U)\,\Phi([\gamma]_V^{-1})$$

$$= [\gamma]_X\, \Phi([\gamma]_V)^{-1}$$

$$= [\gamma]_X \left([\gamma]_X\right)^{-1}$$

$$= [c_p]_X$$

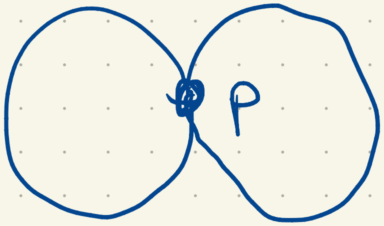
$$\mathcal{C} = \{ [\gamma]_U [\gamma]_V^{-1} : \gamma \text{ is a loop in } U \cap V \text{ with base point } p \}$$

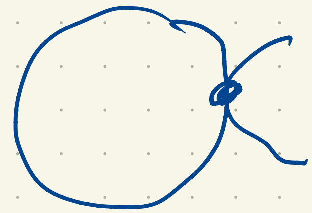
$$\ker \Phi \supseteq \mathcal{C} \implies \ker \Phi \supseteq \overline{\mathcal{C}}$$

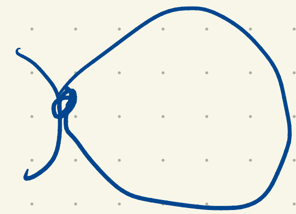
Siefert Van Kampen Theorem:

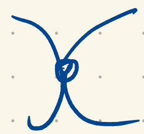
Φ is surjective and $\ker \Phi = \overline{\mathcal{C}}$

so $\pi_1(X, p) \cong \pi_1(U, p) * \pi_1(V, p) / \overline{\mathcal{C}}$.



U : 

V : 

$U \cap V$:  $\simeq$

$\pi_1(U, p) \simeq \mathbb{Z}$

$\pi_1(V, p) \simeq \mathbb{Z}$

$\pi_1(U \cap V, p)$ is trivial.

$\pi(X, p) \simeq \pi_1(U, p) * \pi(V, p) / \overline{\mathcal{C}}$

$\mathcal{C} = \{ [\gamma]_U \cdot [\gamma]_V^{-1} : [\gamma]_{U \cap V} \}$

$\Downarrow$

$[c_p]_{U \cap V}$

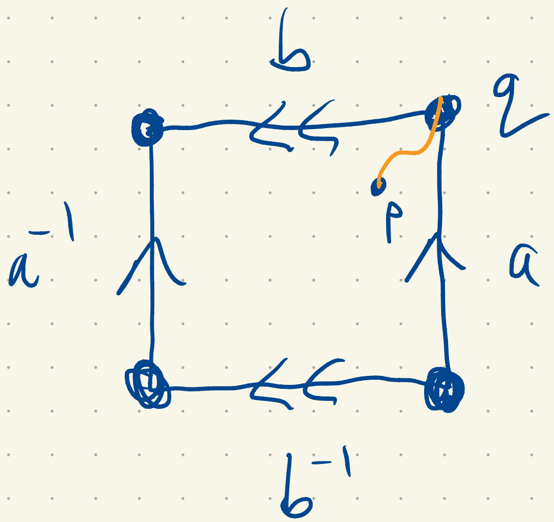
$$C = \{ \underbrace{[c_p]_U \cdot [c_p]_V^{-1}}_{\text{identity in the free product}} \}$$

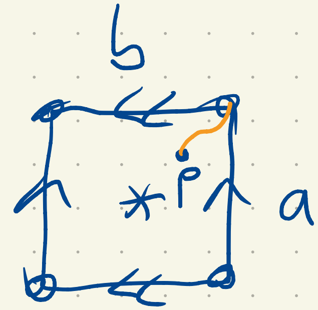
$$\overline{C} = \{ 1_{f,p} \}$$

$$\pi_1(X,p) \cong \mathbb{Z} * \mathbb{Z}$$

Whenever $\pi_1(U \cap V, p)$ is trivial,

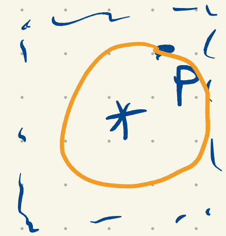
$$\pi_1(X,p) \cong \underline{\pi_1(U,p)} * \underline{\pi_1(V,p)}$$

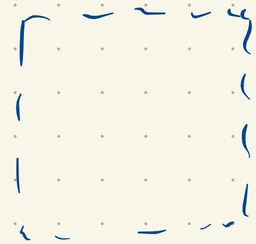


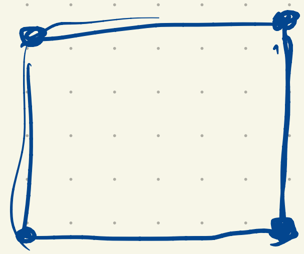
U: 

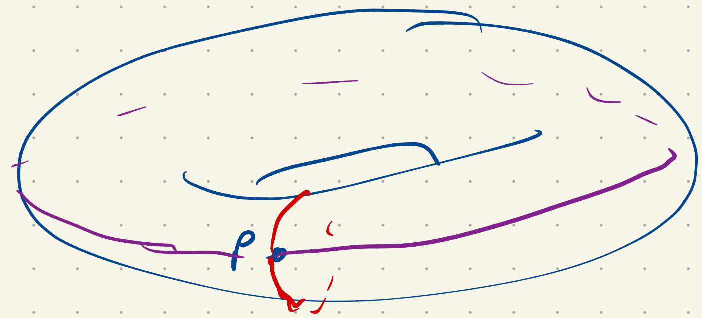
$hah^{-1} = \hat{a}$

$\tilde{b}$

$U \cap V$: 

V: 





$\pi_1(U, q) \cong \mathbb{Z} * \mathbb{Z}$

$$\pi_1(U,p) \simeq \pi_1(U,q)$$

$$\pi_1(U \cap V, p) \simeq \mathbb{Z}$$

Because $\pi_1(V,p)$ is trivial

$$\pi_1(X,p) \simeq \pi_1(U,p) \big/ \overline{\pi_1(U\cap V,p)}$$

$$\pi_1(U,p) \downarrow \mathbb{Z} * \mathbb{Z}$$

$$\hat{a}\hat{b}\hat{a}^{-1}\hat{b}^{-1} \sim id$$

$$\hat{a}\hat{b} \sim \hat{b}\hat{a}$$

$\to$ $\mathbb{Z} * \mathbb{Z}$, quotient that is abelian.

$$\langle S \mid \underset{\sim}{R} \rangle \qquad \langle S \rangle / \overline{R}$$

$$\pi_1(U, p) \sim \langle S_1 \mid R_1 \rangle$$

$$\pi_1(V, p) \sim \langle S_2 \mid R_2 \rangle$$

$$\pi_1(U \cap V, p) \sim \langle S_3 \mid R_3 \rangle$$

$$\pi_1(U, p) * \pi_1(V, p) / \overline{C} \sim \langle S_1 \cup S_2 \mid R_1 \cup R_2 \cup R' \rangle$$

$s \in S_3$ write $s = s_1$ in $\pi_1(U, p)$

$s = s_2$ in $\pi_1(V, p)$

R' consists of s_1, s_2^{-1}