G_1, G_2 $\qquad$ $\underbrace{G_1 * G_2}$

$\underbrace{(g_1, 1_2}, \ldots, g_n)$ word.

Every word is related to a unique reduced word,

$$\underset{\text{reduced}}{r : \underset{\text{words}}{\mathcal{W}} \longrightarrow R}$$

(every word is clearly related to at least one reduced word)

1) if w is reduced $r(w) = w$

2) if $w \sim w'$ $r(w) = r(w')$

$w \sim v$
$w \sim v'$
reduced

$v = r(v) = r(w) = r(v') = v'$

reduced $\longrightarrow v = (h_1, h_2, \ldots, h_n)$ $\qquad \odot$

$$v \odot () = v$$

$$g \in G_\alpha$$

$$v \odot (g) = \begin{cases} () & n = 0,\ 1_\alpha \\ (g) & g \neq 1_\alpha \\ (h_1, \ldots, h_{n-1}) & h_n \in G_\alpha \quad h_n g = 1_\alpha \\ (h_1, \ldots, h_{n-1}, h_n g) & h_n \in G_\alpha \quad h_n g \neq 1_\alpha \\ \underline{(h_1, \ldots, h_n)} & h_n \notin G_\alpha \quad g = 1_\alpha \\ (h_1, \ldots, h_n, g) & h_n \notin G_\alpha \quad g \neq 1_\alpha \end{cases}$$

reduced $\qquad$ word

$$v \odot (g_1, \ldots, g_m) = \Big(\big((v \odot g_1) \odot g_2 \big) \odot g_3 \big) \cdots \odot g_m \Big)$$

The result $R(V, W')$ is reduced.

reduced $\quad$ word

$R(V,W) = VW$ if VW is reduced.

If $W \sim W'$

$$R(V,W) = R(V,W')$$

It's enough to show this if W, W' are related by a single elementary reduction.

$$\left[\begin{array}{l} W = \cdots, g\, g', \cdots \\ W' = \cdots, g, g', \cdots \end{array}\right.$$

$$R(V,W) = R(V,W')$$

$W = \ldots, \ldots$

$W' = \ldots, 1_a, \ldots$

$r(W) = R((), W)$

reduced

if W is reduced

then $() \cdot W$ is reduced

so $R((), W) = W$

If $w \sim w'$

$$r(w) = R((), w) = R((), w') = r(w')$$

$g_1 \in G_1$ $\qquad$ $g_1 g_2 \neq g_2 g_1$

$g_2 \in G_2$

$G_1 * G_2$

There is a natural $\underbrace{\text{map}}_{\substack{\downarrow \\ \text{injective} \\ \text{group hom.}}}$ $G_1 \longrightarrow G_1 * G_2$

$$g \xmapsto{\phi_1} g$$

$$\phi_1(g g') = g g'$$
$$= \phi_1(g)\, \phi_1(g')$$

Suppose I have two group homs

$$\psi_1 : G_1 \to H$$

$$\psi_2 : G_2 \to H$$

Want to "merge" to get a map

$$\Phi : G_1 * G_2 \to H.$$

Characteristic Property of Free Product:

Suppose $\psi_i : G_i \to H$ are homs.

Then there exists a unique hom $\Phi : G_1 * G_2 \to H$ such that for each α

$$\begin{tikzcd}
G_1 * G_2 \arrow[dr, bend left, "\Phi"] & \\
G_\alpha \arrow[u, "i_\alpha"] \arrow[r, "\psi_\alpha"'] & H
\end{tikzcd}$$

Sketch: $(g_1, \dots, g_n)$

$$\Phi(g_1 \cdots g_n) = \Phi(g_1) \cdot \dots \cdot \Phi(g_n)$$

$$= \underbrace{\psi_{\alpha_1}(g_1) \cdot \dots \cdot \psi_{\alpha_n}(g_n)}_{\text{no choices}}$$

If Φ exists then it is unique.

$$\Phi((g_1, g_2)) \overset{?}{=} \Phi((g_1 g_2)) \qquad g_1, g_2 \in G_\alpha$$

$$\Phi(g_1 g_2) = \psi_\alpha(g_1 g_2)$$

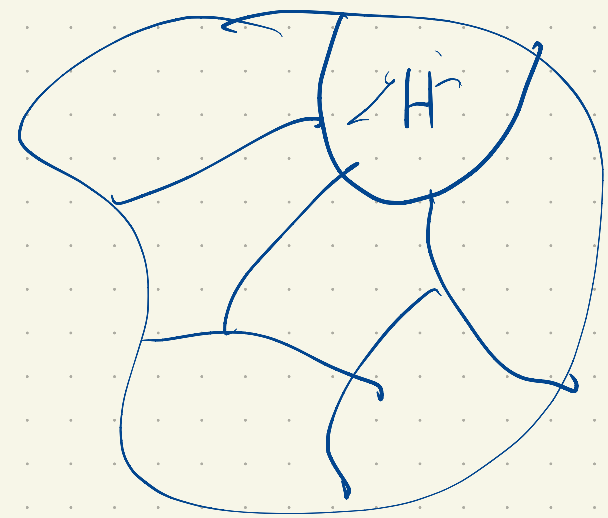
$$= \psi_\alpha(g_1)\psi_\alpha(g_2)$$

$$\Phi((g_1, g_2)) = \Phi(g_1)\Phi(g_2) = \psi_\alpha(g_1)\psi_\alpha(g_2)$$

Normal subgroups $H \subseteq G$

$g^{-1} h g \in H$ whenever $h \in H$

$g^{-1} H g \subseteq H$



$G \longrightarrow \underbrace{G/H}$ $\longrightarrow gH$

$g \longmapsto gH$

$g \in G$ gH

Every kernel of a homomorphism is normal.

ϕ $\qquad$ $k \in \ker \phi$

$$\begin{aligned}\phi(g^{-1} k g) &= \phi(g^{-1}) \phi(k) \phi(g) \\ &= \phi(g^{-1}) \phi(g) \\ &= \phi(g^{-1} g) \\ &= 1 \Rightarrow g^{-1} k g \in \ker \phi\end{aligned}$$

Normal subgroups are precisely the kernels of group homs.

$\{N_\alpha\}_{\alpha \in I}$ $N_\alpha \subseteq G$, normal subgroup

Is $\bigcap_{\alpha \in I} N_\alpha$ normal? Yep!

n $g^{-1} n g \in N_\alpha$ since $n \in N_\alpha$, $\forall \alpha$.

Given $C \subseteq G$, a set, maybe not even a subgroup

we can construct the smallest normal subgroup contains C.

It is the intersection of all normal subgroups contains C.

It's called the normal closure of C, $\overline{C}$.