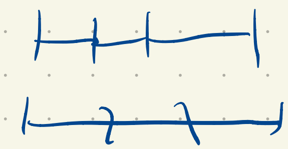
Def: $\alpha_n : I \to S^1$

$$\alpha_n(s) = e^{2\pi i n s}$$

$$\alpha = \alpha_1$$

$(\alpha \cdot \alpha)\cdot \alpha$



Exercise: $[\alpha]^n = [\alpha_n]$

Thm: $\pi_1(S^1, 1)$ is infinite cyclic with generator $[\alpha]$

(equivalently $\mathbb{Z} \to \pi_1(S^1, 1)$, $n \mapsto [\alpha]^n$ is a group iso)

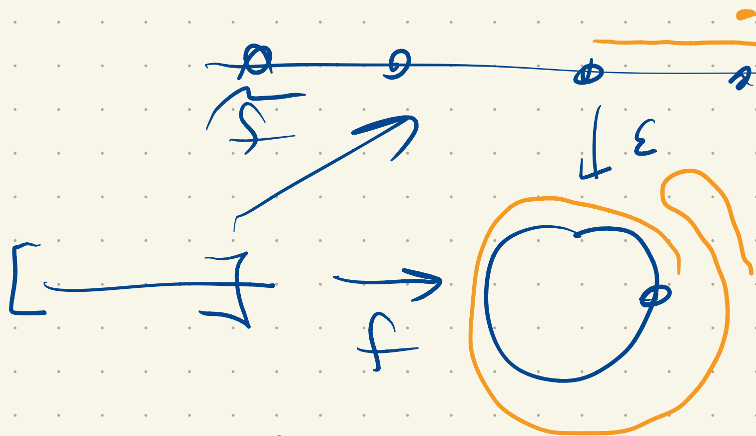
Pf: Define $j : \mathbb{Z} \to \pi_1(S^1, 1)$ by $j(n) = [\alpha]^n$.

Observe $j(n+m) = [\alpha]^{(n+m)} = [\alpha]^n [\alpha]^m = j(n)\, j(m)$.

So j is a group homomorphism.

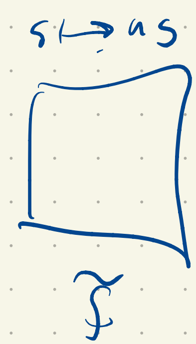
To see that j is surjective consider $[f] \in \pi_1(S^1, 1)$,

Let $\tilde{f}$ be a lift of f starting at 0.



Let $n = \tilde{f}(1)$.

Let $H(s,t) = (1-t)\tilde{f}(s) + t\,ns$



Observe that H is a path homotopy.

Moreover $\varepsilon \circ H$ is a path homotopy from f to $\left(s \mapsto \varepsilon(ns) \right)$ $e^{2\pi i n s}$

α_n. So $[f] = j(n)$.

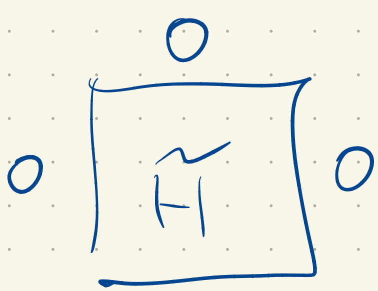
To establish injectivity suppose $j(n) = 1 = [c_1]$
$= [\alpha_0]$.

We need to show $n = 0$. Let H be a path homotopy from $j(n)$ to $[\alpha_0]$, i.e. from $[\alpha_n]$ to $[\alpha_0]$.

By homotopy lifting we obtain

$\tilde{H} : I \times I \to \mathbb{R}$ with $\varepsilon \circ \tilde{H} = H$

and such that $\tilde{H}(0,0) = 0$.



Using the fact that constants lift to constants we obtain $\tilde{H} = 0$ on three sides.

In particular $\tilde{H}(1,0) = 0$.

But $\tilde{H}(s,0)$ is a lift of α_n starting at 0. Hence $\tilde{H}(s,0) = ns$. Since $\tilde{H}(1,0) = 0$

We conclude $n = 0$.

Fundamental Groups from pieces

Let G_1 and G_2 be groups $G_1 \cap G_2 = \phi$ for simplicity

$\coprod_{\alpha \in 1,2} G_\alpha$ A word in $G_1 \cup G_2$ is a finite tuple, possibly empty, $(g_1, \ldots, g_n)$ with $g_i \in G_1 \cup G_2$.

We have a product on words $(g_1, \ldots, g_n) \circ (h_1, \ldots, h_m)$

$= (g_1, \ldots, g_n, h_1, \ldots, h_m)$.

$g, g' \in G_1$

$g_1 g_2 \cdots g_n$

$(g)\ (g') = (gg')$

$\downarrow$

(g, g') $\qquad (\rule{2cm}{0.4pt}) \ (1_\alpha)$

$(\rule{2cm}{0.4pt},\ 1_\alpha)$

Elementary reductions

$$(g_1, \dots, g_{i-1}, 1_\alpha, g_{i+1}, \dots, g_n) \longrightarrow (g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$$

If $g_i, g_{i+1} \in G_\alpha$

$$(g_1, \dots, g_{i-1}, g_i, g_{i+1}, \dots, g_n) \longrightarrow (g_1, \dots, g_{i-1}, g_i g_{i+1}, \dots, g_n)$$

We say words W, W' are related if there is a finite sequence

$$W = W_1, W_2, \dots, W_m = W'$$

such that for each j there is an elementary reduction taking W_j to W_{j+1} or vice-versa.

Exercise: This is an equivalence relation.

Def: $G_1 * G_2$ (the free product of G_1 with G_2)

is the set of equivalence classes of words

in $G_1 \cup G_2$ under this equiv. relation.

We define a product on $G_1 * G_2$ by

$$[w_1] \cdot [w_2] = [w_1 w_2].$$

$$w_1' \qquad w_2$$

Exercise: this is well defined.

$$[(1_{G_1})] \stackrel{?}{=} [()]$$

The identity element is $[()]$

If $W = (g_1, \dots, g_n)$

$$[W]^{-1} = [(g_n^{-1}, \dots, g_1^{-1})]$$

$$[W][(g_n^{-1}, \dots, g_1^{-1})]$$

$$= [(g_1, \dots, \underbrace{g_n, g_n^{-1}}, \dots, g_1^{-1})]$$

$$= [(g_1, \dots, g_n g_n^{-1}, \dots, g_1^{-1})]$$

$$= [(g_1, \dots, 1_\alpha, \dots, g_1^{-1})]$$

$$= [(g_1, \dots, g_{n-1}, g_{n-1}^{-1}, \dots, g_1^{-1})]$$

$\vdots$

$$= \left[(g_1, g_1^{-1}) \right]$$

$$= \left[(1_\alpha) \right]$$

$$= \left[(\,) \right].$$

Associativity. $\left([w_1][w_2] \right)[w_3] = \left([w_1 w_2] \right)[w_3]$

$$= \left[(w_1 w_2) w_3 \right]$$

$$= \left[w_1 w_2 w_3 \right]$$

If $g_1 \in G_1$ and $g_2 \in G_2$ $g_1 \neq 1d$ $g_2 \neq id$,

is $g_1 g_2 = g_2 g_1$

No!

We say a word is <u>reduced</u> if it contains no identity elements and no two adjacent entries come from the same group.

Claim: every word is related to a <u>unique</u> reduced word.

Plan: I'm going to build

$$r : \mathcal{W} \to R$$

words $\uparrow$ $\qquad$ $\uparrow$ reduced words

such that $\quad$ 1) $r(w) = w$ if w is reduced

2) $r(w) = r(w')$ if $w \sim w'$

If I do this suppose w is related to the reduced words v, v'.

Then $v' = r(v') = r(w) = r(v) = v$