$r(x)$

$r : \mathbb{R}^2 \to S^1$

$r(x_1) = z_1$

$x_1 \to x$

$z_1 \to z$

$x_1 \to x$

$\boxed{r(x_1) \to z}$

Job: $z = r(x)$

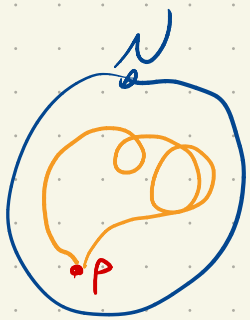
$$r(x) = f(x) + t(x)\big(x - f(x)\big)$$

$$z_1 = f(x_1) + t(x_1)\big(x_1 - f(x_1)\big)$$

$$z = f(x) + \tau\big(x - f(x)\big)$$

$$z_1 - f(x) = t(x_1)\big(x_1 - f(x_1)\big)$$

$\pi_1(S^n, p) \quad n \geq 2$



Claim: for a manifold M^n with $n \geq 2$

and a path f from some p to some r

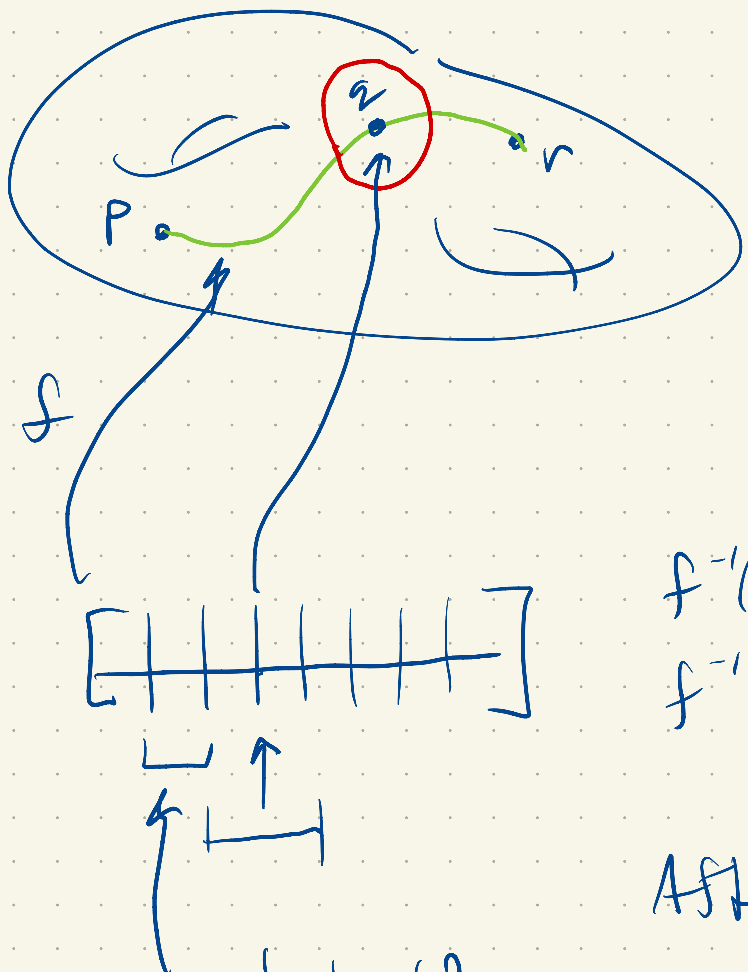
if $q \in M$ and $q \neq p$, $q \neq r$ then

f is path homotopic to a path that does not contain q.

U, open about q

and U is homeomorphic to $\mathbb{R}^n$.

$V = M \setminus \{q\}$

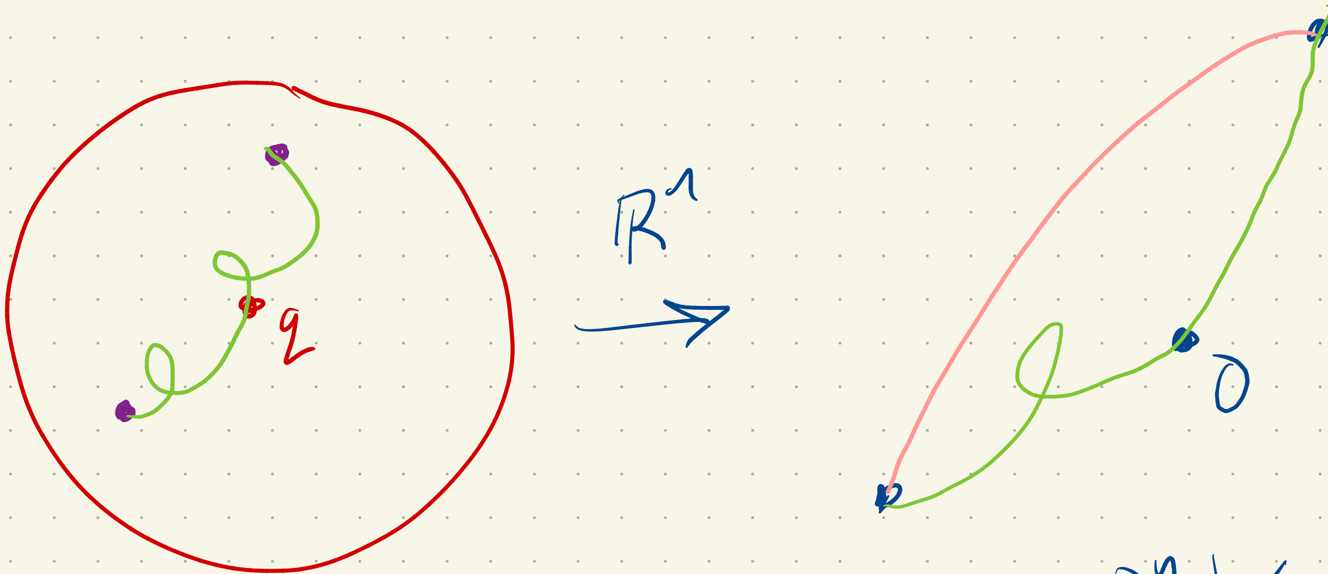


$f^{-1}(U)$

$f^{-1}(V)$

sent to U.

After amalgamation, no subinterval endpoint is q.



$\mathbb{R}^n \setminus \{0\}$

is path connected

Want: If $\varphi : X \to Y$ is a homotopy equivalence

then $\varphi_* : \pi_1(X, p) \to \pi_1(Y, \varphi(p))$ is a group isomorphism

Spirit: Let ψ be a homotopy inverse

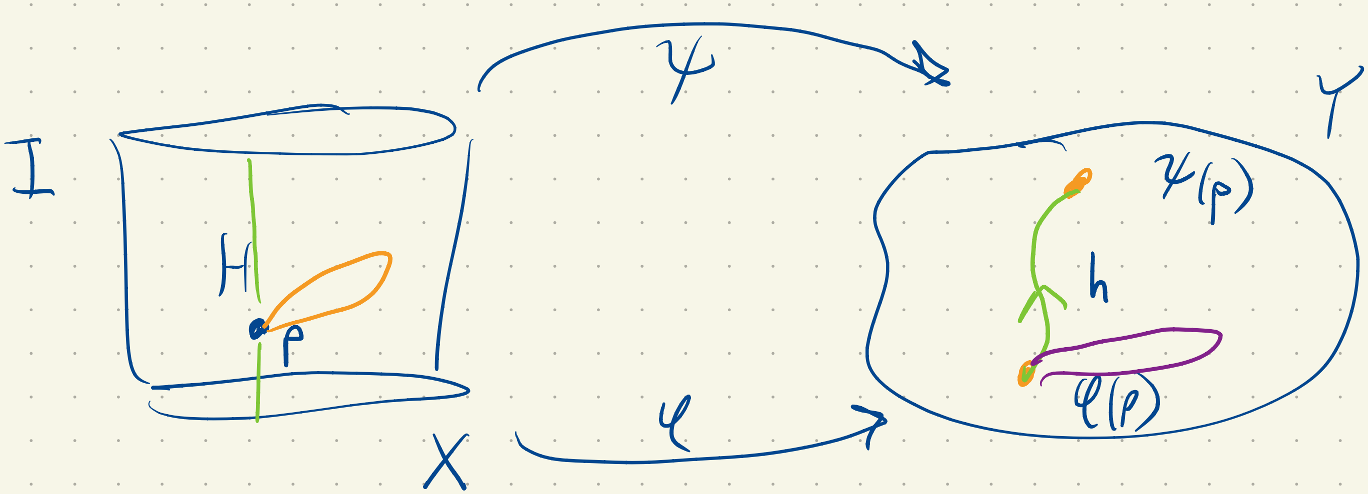
So $\quad \psi \circ \varphi \sim \mathrm{id}_X$

$$\underbrace{\psi_* \circ \varphi_*} \;\cancel{=}\; (\mathrm{id}_X)_*$$

$(\psi \circ \varphi)(p) \neq p$ in general, Technical Lemma: Suppose φ and $\psi : X \to Y$ are homotopic with homotopy H. Fix $p \in X$ and let $h(t) = H(p,t)$ $(p \in X)$.

Then $\Phi_h : \pi_1(Y, \varphi(p)) \to \pi_1(Y, \psi(p))$ satisfies

$$\begin{tikzcd}
 & \pi_1(Y, \varphi(p)) \arrow[d, "\Phi_h"] \\
\pi_1(X, p) \arrow[ur, "\varphi_*"] \arrow[r, "\psi_*"'] & \pi_1(Y, \psi(p))
\end{tikzcd}$$



Pf: Let f be a loop based at p,

We wish to show $\psi_*[f] = \Phi_h(\underline{\varphi_*[f]})$,

This is equivalent to showing

$$\psi \circ f \sim_p \bar{h} \cdot \varphi \circ f \cdot h$$

or equivalently

$$h \cdot (\psi \circ f) \sim_p (\varphi \circ f) \cdot h.$$

Consider $F(s,t) = H(f(s), t)$:

$$\begin{tikzcd}
\bullet \arrow[r, "\psi\circ f"] \arrow[d, "h"'] \arrow[dr, phantom, "F"] & \bullet \arrow[d, "h"] \\
\bullet \arrow[r, "\varphi\circ f"'] & \bullet
\end{tikzcd}$$

That $h\cdot(\psi\circ f) \sim_p (\varphi\circ f)\cdot h$

is a consequence of

the Square Lemma,

$$\begin{tikzcd}
\bullet \arrow[r, "i"] \arrow[d, "h"'] & \bullet \arrow[d, "j"] \\
\bullet \arrow[r, "k"'] & \bullet
\end{tikzcd}$$

$$h\cdot i \sim_p k\cdot j$$

Thm: If $\varphi : X \to Y$ is a homotopy equivalence then

$$\varphi_* : \pi_1(X,p) \to \pi_1(Y, \varphi(p)) \text{ is a group}$$

isomorphism.

Pf: It suffices to show that φ_* is bijective.

Let ψ be a homotopy inverse so $\psi \circ \varphi \sim \mathrm{Id}_X$.

Consider

$$\begin{tikzcd}
\pi_1(X,p) \arrow[r, "(\mathrm{Id}_X)_*"] \arrow[dr, "(\psi\circ\varphi)_*"'] & \pi_1(X,p) \arrow[d, "\Phi_h"] \\
 & \pi_1(X, \psi(\varphi(p)))
\end{tikzcd}$$

Hence $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ is a group isomorphism.

In particular, ψ_* is surjective (and φ_* is injective).

Note $\psi_* : \pi_1(Y, \varphi(p)) \longrightarrow \pi_1(X, \psi(\varphi(p)))$.

Consider

$$\begin{tikzcd}
\pi_1(Y, \varphi(p)) \arrow[r, "Id_Y"] \arrow[dr, "(\varphi \circ \psi)_*"'] & \pi_1(Y, \varphi(p)) \arrow[d, "\Phi_K"] \\
 & \pi_1(Y, \varphi(\psi(\varphi(p))))
\end{tikzcd}$$

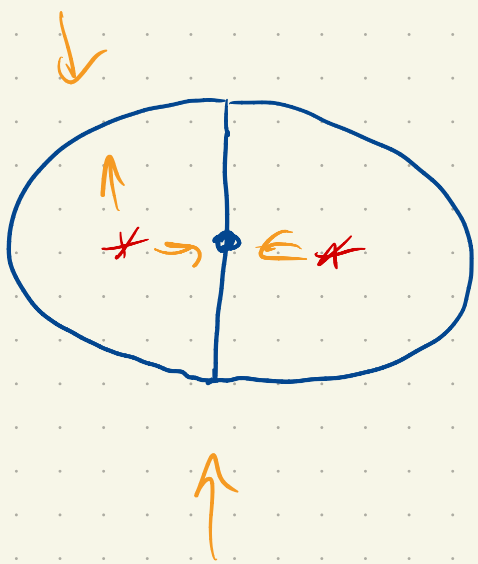
By the same argument, $(\varphi \circ \psi)_*$ is bijective,

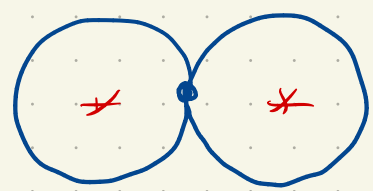
so ψ_* is injective. $\varphi_* \circ \psi_* \longrightarrow \varphi_{*, \psi(\varphi(p))}$

Hence ψ_* is bijective. But then, returning to

$$\psi_* \circ \varphi_* = (\mathrm{Id}_X)_*$$

we conclude φ_* is bijective.





θ-space is a

deformation retract

$$\left.\begin{array}{l} r \circ i_A = id \\ i_A \circ r \simeq id \end{array}\right]$$ deformation retract

$A \subseteq X$ is homotopy equivalent to X