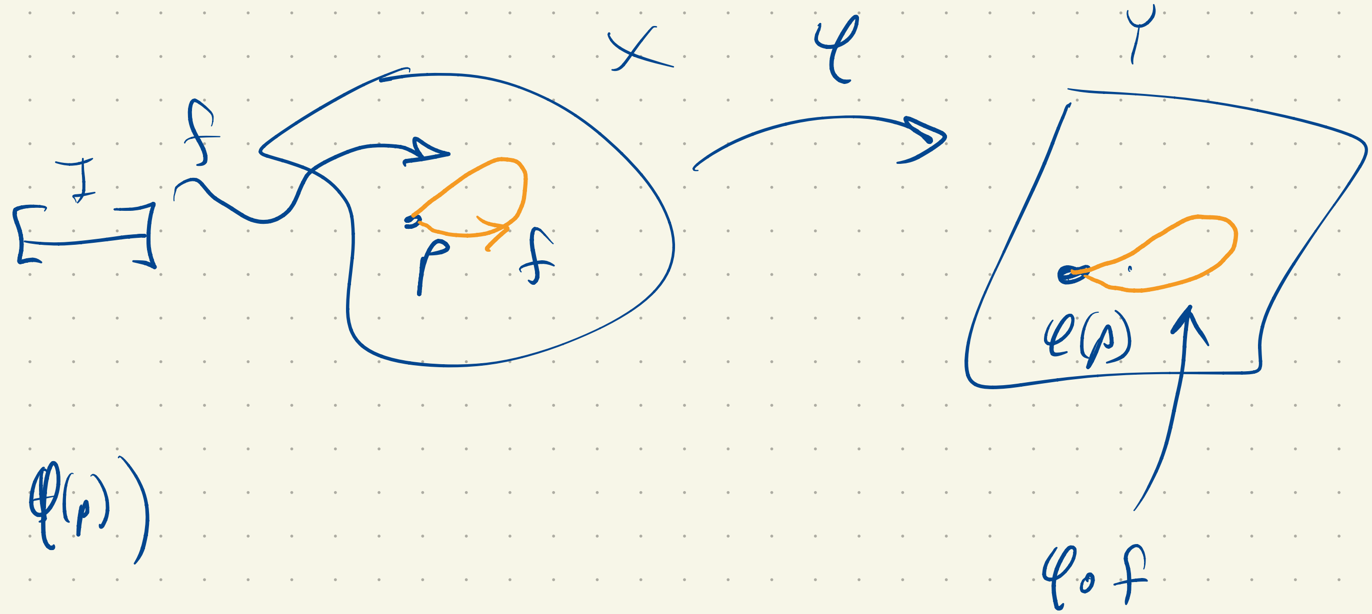
$X \xrightarrow{\varphi} Y$

$p \in X$



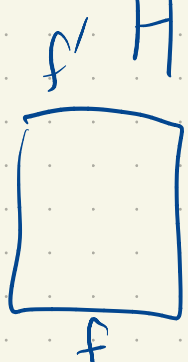
φ_*

$\pi_1(X,p) \longrightarrow \pi(Y, \varphi(p))$

$$\underline{\varphi_*([f]) := [\varphi \circ f]}$$

$f' \sim_p f$ $\qquad\qquad$ $\varphi \circ f' \sim_p \varphi \circ f$?

H $\qquad\qquad$ $\varphi \circ H$



Claim φ_* is a group hom.

$$\varphi_*\left([f_1]\cdot[f_2]\right) = \varphi_*\left([f_1\cdot f_2]\right)$$

$$= \left[\varphi\circ(f_1\cdot f_2)\right]$$

$$= \left[(\varphi\circ f_1)\cdot(\varphi\circ f_2)\right]$$

$$= [\varphi\circ f_1]\cdot[\varphi\circ f_2]$$

$$= \varphi_*([f_1])\cdot\varphi_*([f_2])$$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi \circ \varphi"'] & Y \arrow[r, "\psi"] & Z \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(X, p) \arrow[r, "\varphi_*"] \arrow[rr, bend right, "\psi_* \circ \varphi_*"'] & \pi_1(Y, \varphi(p)) \arrow[r, "\psi_*"] & \pi_1(Z, \psi(\varphi(p))) \end{tikzcd}$$

$$\psi_* \circ \varphi_* = (\psi \circ \varphi)_*$$

$$\psi_*\left(\varphi_*([f])\right) = \psi_*\left([\varphi\circ f]\right)$$

$$= [\psi\circ\varphi\circ f]\,.$$

$$= (\psi\circ\varphi)_*\left([f]\right)$$

$Id : X \to X$

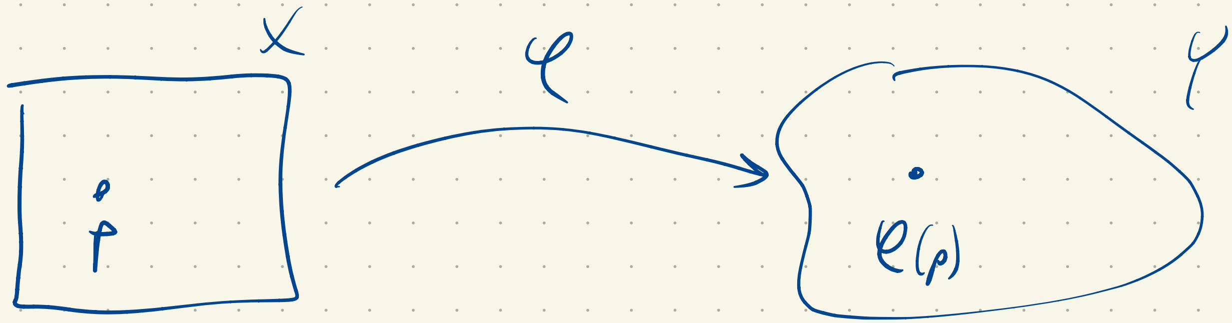
$Id_*\,\pi_1(X,p) \to \pi_1(X,p)$

$Id_*[f] = [Id\circ f]$

$= [f]$ ☺

If X is homeomorphic to Y via some φ

then $\pi_1(X,p)$ is group isomorphic to $\pi_1(Y,\varphi(p))$



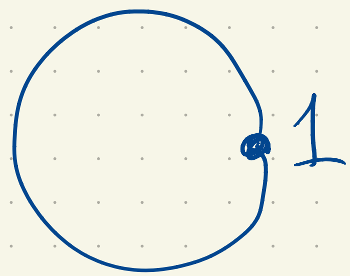
$$\varphi_* : \pi_1(X,p) \to \pi_1(Y,\varphi(p))$$

$$(\varphi^{-1})_* : \pi_1(Y,\varphi(p)) \to \pi_1(X,p)$$

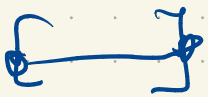
$$(\varphi^{-1})_* \circ (\varphi)_* = (\varphi^{-1}\circ\varphi)_* = (id_X)_* = id_{\pi_1(X,p)}.$$

$$\pi_1(S^1, 1) \sim \mathbb{Z}$$

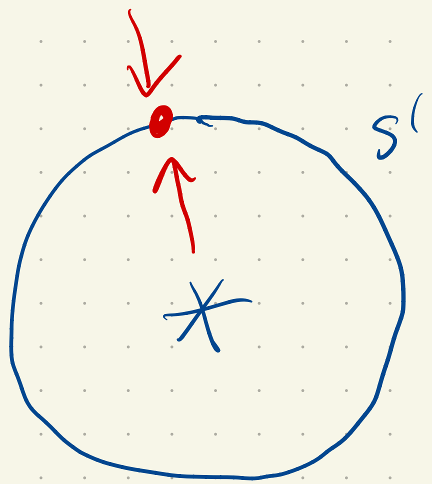
$\hookrightarrow \in \mathbb{C}$



$\omega_1(s) = e^{2\pi i s}$



Claim $\mathbb{R}^{2,*} = \mathbb{R}^2 \setminus \{0\}$ has nontrivial fund. group



$r : \mathbb{R}^{2,*} \to S^1$

$r(x) = \dfrac{x}{|x|}$

$i : S^1 \to \mathbb{R}^{2,*}$

$r \circ i = \mathrm{id}_{S^1}$

We call r a retraction and S^1 a retract of $\mathbb{R}^{2,*}$

$X, \quad A \subseteq X \qquad \left.\begin{array}{l} r: X \to A \\ r(a) = a \quad \forall a \in A \end{array}\right]$ retract.

$r \circ i = id_{S^1}$

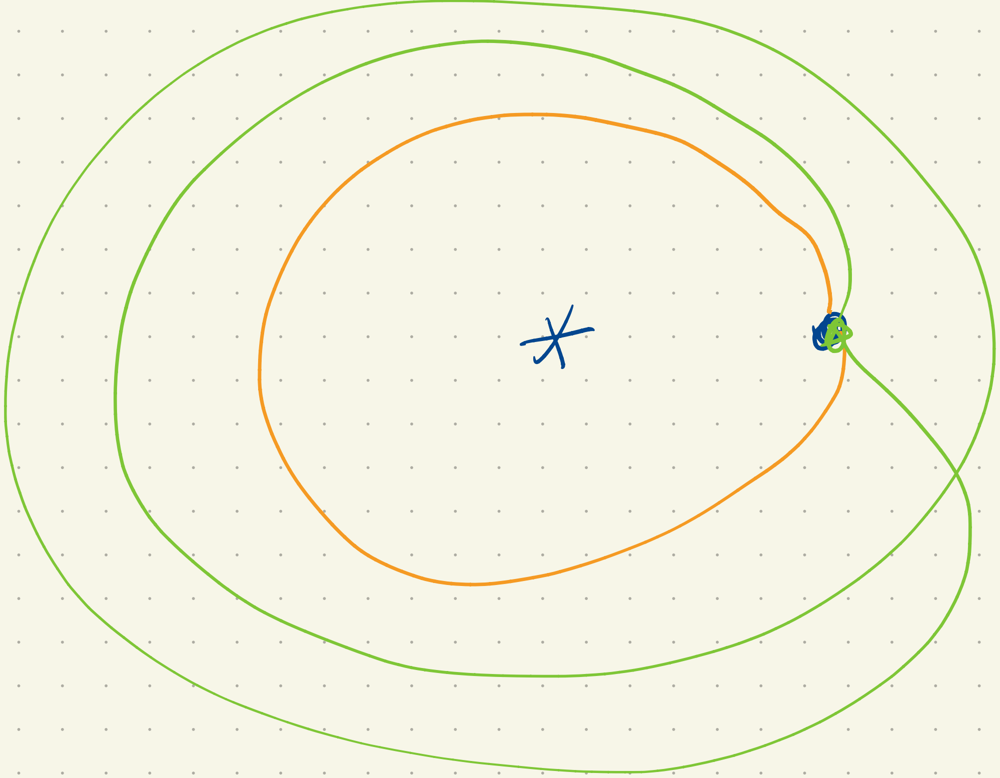
$\downarrow$

$$\underbrace{r_* \circ i_*}_{} = id_{\pi_1(S^1, 1)}$$

surjective injective bijection

$$\boxed{\pi_1(\mathbb{R}^{2,*}, 1)}$$

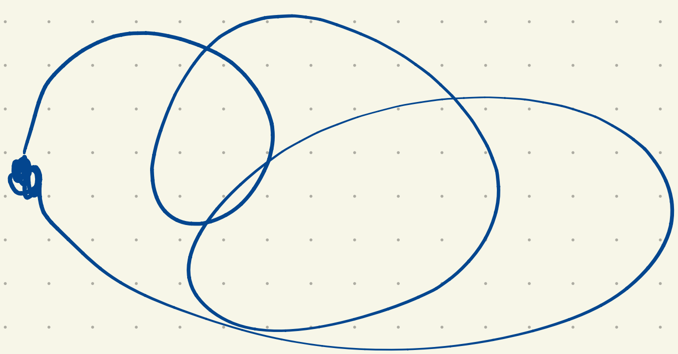
1) it contains a subgroup isomorphic to $\mathbb{Z}$.



$\pi_1(\mathbb{R}^{2*}, \underset{\in\mathbb{C}}{1}) \simeq \mathbb{Z}$

In fact if X is homotopy equivalent to Y via φ

then $\pi_1(X, p) \simeq \pi_1(Y, \varphi(p))$



Facts: 1) $\pi(S^n, p)$ $\quad (p \in S^n)$

is trivial if $n \geq 2$.

(S^n is simply connected)

2) Given spaces X_1 and X_2

$p_1 \in X_1 \qquad p_2 \in X_2$

$$\pi_1(X_1 \times X_2, (p_1, p_2)) \simeq \underbrace{\pi_1(X_1, p_1) \times \pi_1(X_2, p_2)}$$

$\hookrightarrow$ direct product of groups

$G_1, G_2 \qquad (g_1, g_2)$

$G_1 \times G_2$

$T^2 = S^1 \times S^1 \qquad \pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$

$= \mathbb{Z}^2$

$\mathbb{Z}^2 \sim \mathbb{Z}^3$

$T^n = S^1 \times \cdots \times S^1 \qquad \pi(T^n) = \mathbb{Z}^n$

$p_i : X_1 \times X_2 \to X_i$

Claim: $p : \pi_1(X_1 \times X_2, (p_1, p_2)) \to \pi_1(X_1, p_1) \times \pi_1(X_2, p_2)$

defined by $p([f]) = ([p_1 \circ f], [p_2 \circ f])$

is a group isomorphism. $= (p_{1,*}[f], p_{2,*}[f])$

That ϕ is a homomorphism is easy.

To see that P is surjective let $[f_1] \in \pi_1(X_1, p_1)$

$[f_2] \in \pi_1(Y_2, p_2)$.

Consider $f = f_1 \times f_2$. $\left(f(s) = (f_1(s), f_2(s)) \right)$

Then $P([f]) = \left([p_1 \circ f], [p_2 \circ f] \right)$

$= \left([f_1], [f_2] \right)$.

To see that P is injective suppose

$P([f]) = id$.

Recall $P([f]) = ([P_1 \circ f], [P_2 \circ f])$

and have $[P_1 \circ f] = [c_{p_1}]$

$[P_2 \circ f] = [c_{p_2}]$

Let H_1 be a path homotopy from $P_1 \circ f$ to c_{p_1}

and similarly for H_2.

Define $H(s,t) = (H_1(s,t), H_2(s,t))$.

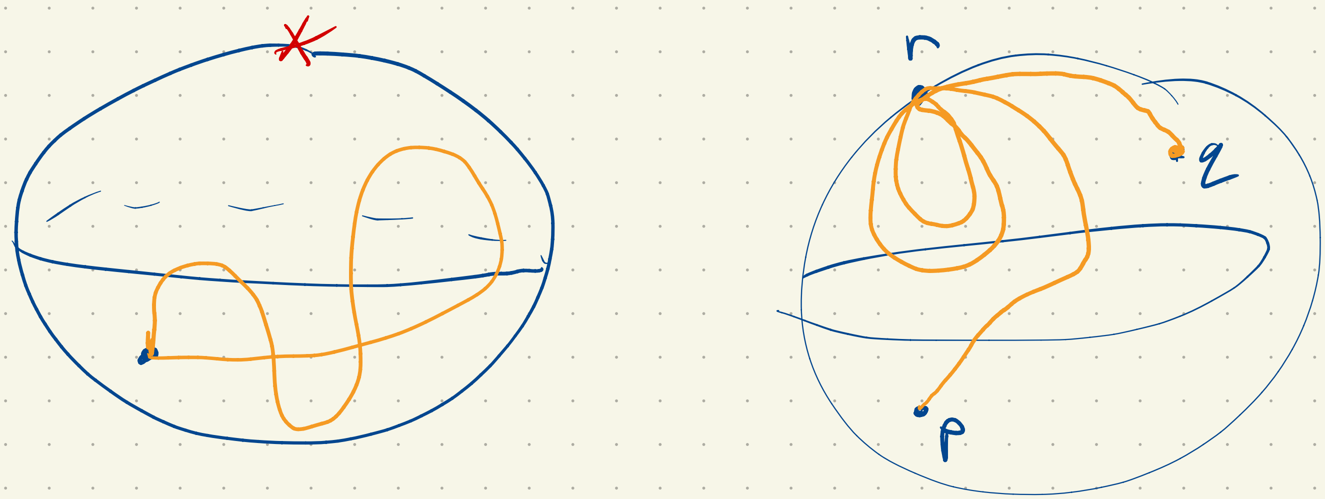
Then $H(s,0) = (H_1(s,0), H_2(s,0))$

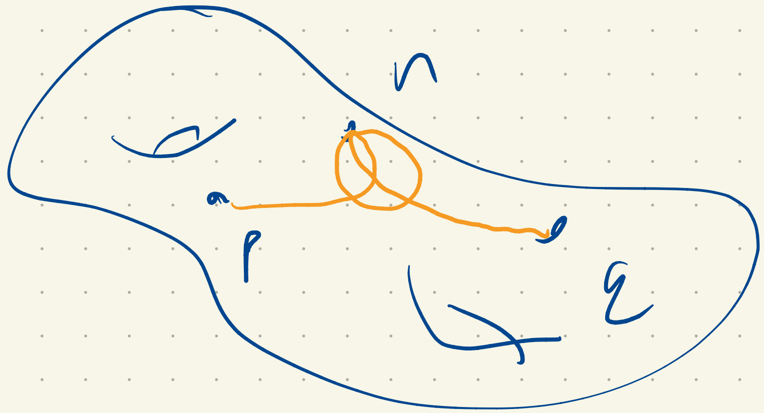
$$= \left(p_1 \circ f(s),\ p_2 \circ f(s) \right)$$

$$= f(s)$$

and $H(s,1) = (p_1, p_2)$ for all s,

So $f \sim_p c_{(p_1, p_2)}$.




r
p
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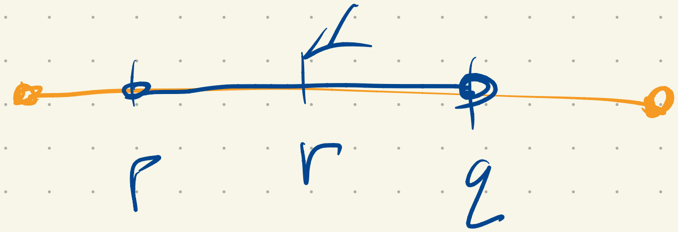
p
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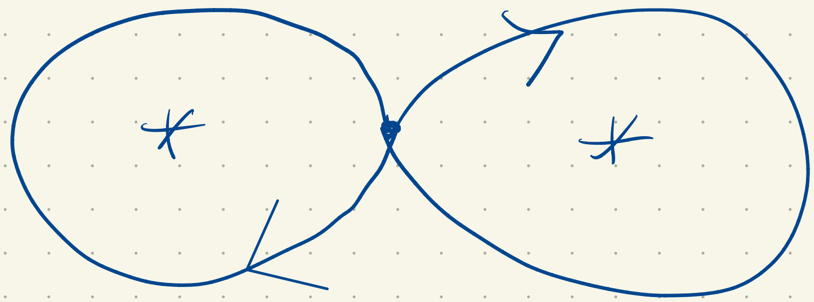

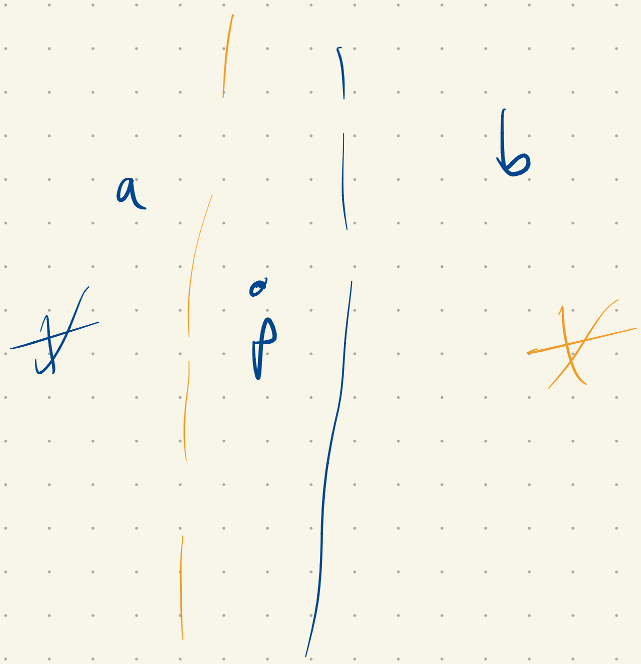
a
p
b