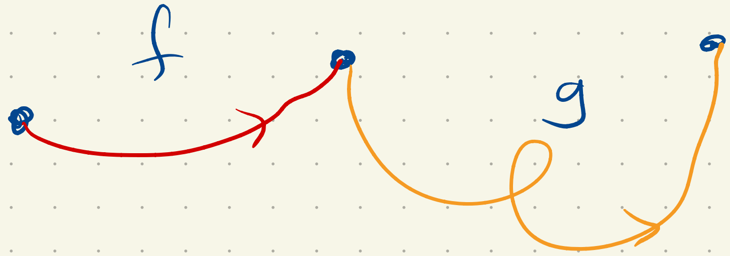
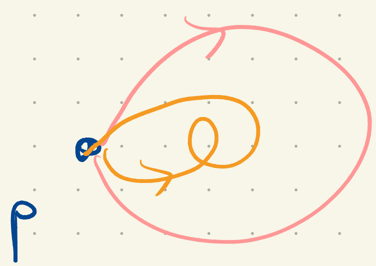


$f \cdot g$

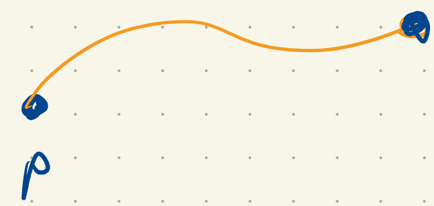
$$[f] \cdot [g] := [f \cdot g]$$

$$f_1 \simeq f_2 \quad g_1 \simeq g_2 \Rightarrow f_1 \cdot g_1 \simeq f_2 \cdot g_2$$

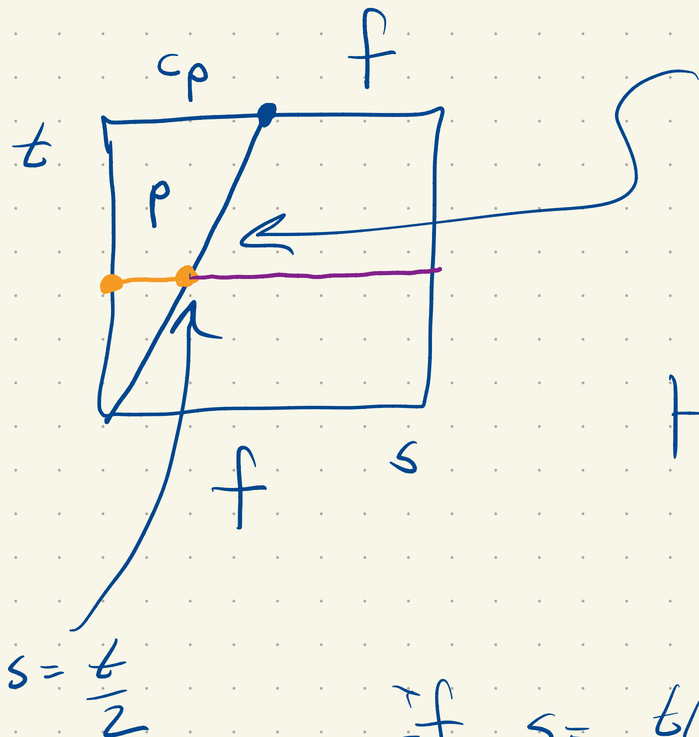


$[f] \cdot [g] \qquad \pi_1(X, p)$

Prop: X a top space, $p \in X$, f a path in X, $f(0) = p$ $\Rightarrow$ $[c_p] \cdot [f] = [f]$



Pf:



$t = 2s$

$$H(s,t) = \begin{cases} p & 0 \le s \le t/2 \\ f\left(\frac{s - t/2}{1 - t/2}\right) & t/2 \le s \le 1 \end{cases}$$

If $s = t/2$

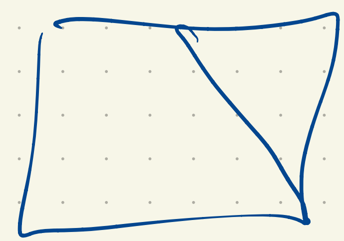
$$H(s,t) = H(t/2, t) = f\left(\frac{t/2 - t/2}{\;}\right)$$

$= f(0)$

$= p$

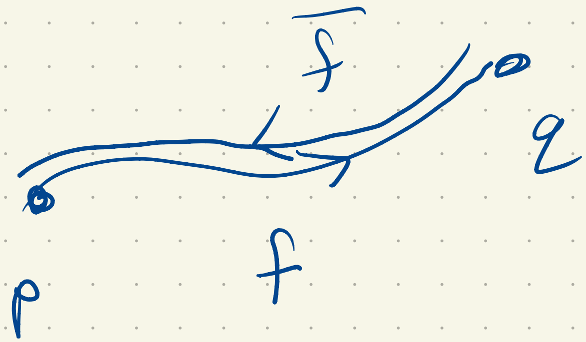
Similarly if f ends at q

$[f]\cdot[e_q] = [f]$



$\pi(X,p)$ identity will be $[c_p]$

$[c_p]\cdot[f] = [f]\cdot[c_p] = [f]$

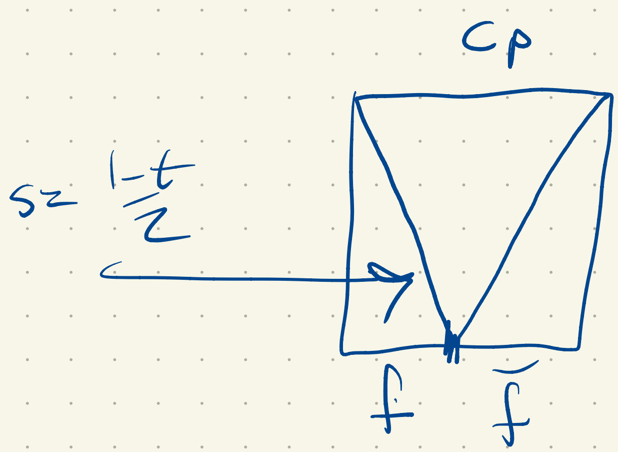


$$\bar{f}(s) = f(1-s)$$

Lemma: Let f be a path in X from p to q.

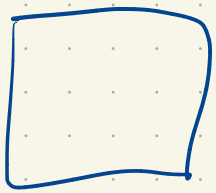
Then

$$\begin{array}{l} [f][\bar{f}] = [c_p] \\ \quad \| \\ \left[f \cdot \bar{f} \right] \end{array}$$



$$H(s,t) = \begin{cases} f(2s) & 0 \le s \le \frac{1-t}{2} \\ f\left(\frac{1-t}{2}\right) & \frac{1-t}{2} \le s \le \frac{1+t}{2} \\ f(2(1-s)) & \frac{1+t}{2} \le s \le 1 \end{cases}$$

Exercise: verify we can apply pasting.

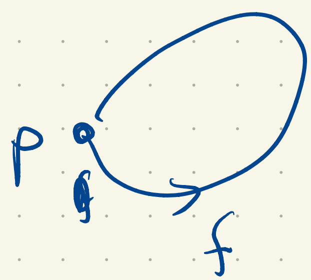


$s \geq \frac{1}{8}$

$$\underbrace{s - \left(\frac{1-t}{2}\right) \leq 0}$$

$g(s,t) =$

$g^{-1}\left((-\infty, 0]\right)$



$[f][\bar{f}] = [c_p]$

$[\bar{f}][f] \overset{?}{=} [c_p]$

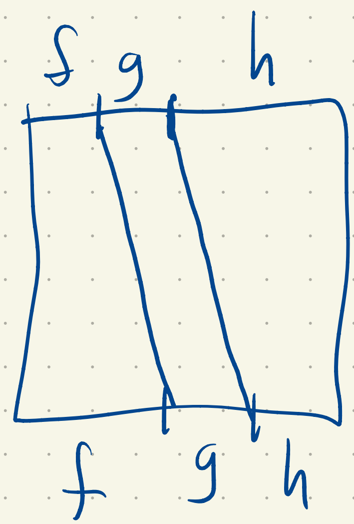
$[\bar{f}][\bar{\bar{f}}] = [c_p]$

$[\bar{f}][\bar{\bar{f}}] = [c_q]$

$$[f]\cdot([g]\cdot[h]) = ([f]\cdot[g])\cdot[h]$$

$$[f]\cdot\,[g\cdot h] = [f\cdot g]\cdot[h]$$

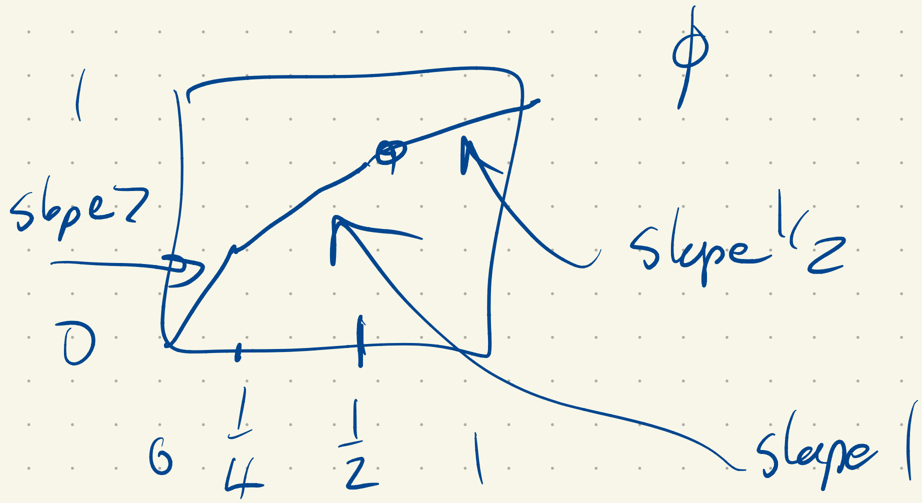
$$[f\cdot\underline{(g\cdot h)}] = [(f\cdot g)\cdot h]$$



In fact

$f\cdot(g\cdot h)$ is a reparameterization

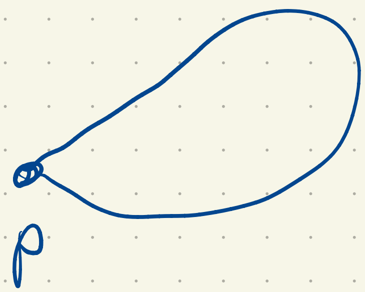
of $(f\cdot g)\cdot h$

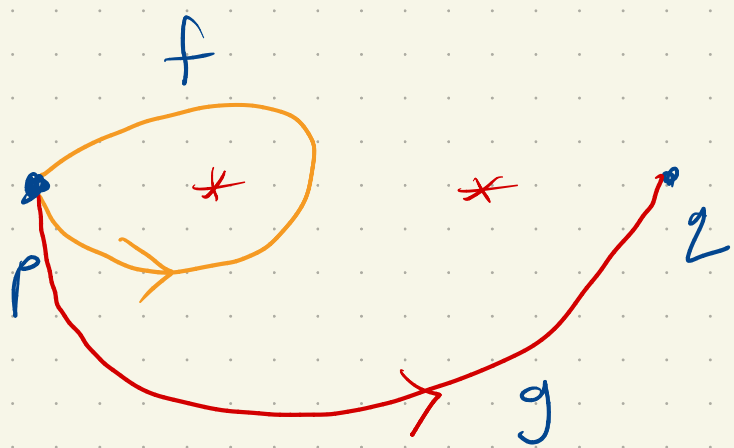


Def: Let X be a top space and $p \in X$. The fundamental group of X based at p is the set of path classes of loops based at p. We denote it by $\pi_1(X,p)$.

We've just seen that $\pi_1(X,p)$ is an algebraic group.

The fundamental group of a space X based at some p can only reveal information about the path component of p.





p in same path comp as q.

How are $\pi_1(X, p)$ and $\pi_1(X, q)$ related?

$$\Phi_g : \pi_1(X,p) \to \pi_1(X,q)$$

$$\Phi_g([f]) = [\bar{g}] \cdot [f][g]$$

Claim: Φ_g is a group isomorphism.

$$\Phi_g([f_1]\cdot[f_2]) = \Phi_g([f_1 \cdot f_2])$$

$$= [\bar{g}][f_1 \cdot f_2] \cdot [g]$$

$$= [\bar{g}][f_1 \cdot c_p \cdot f_2] \cdot [g]$$

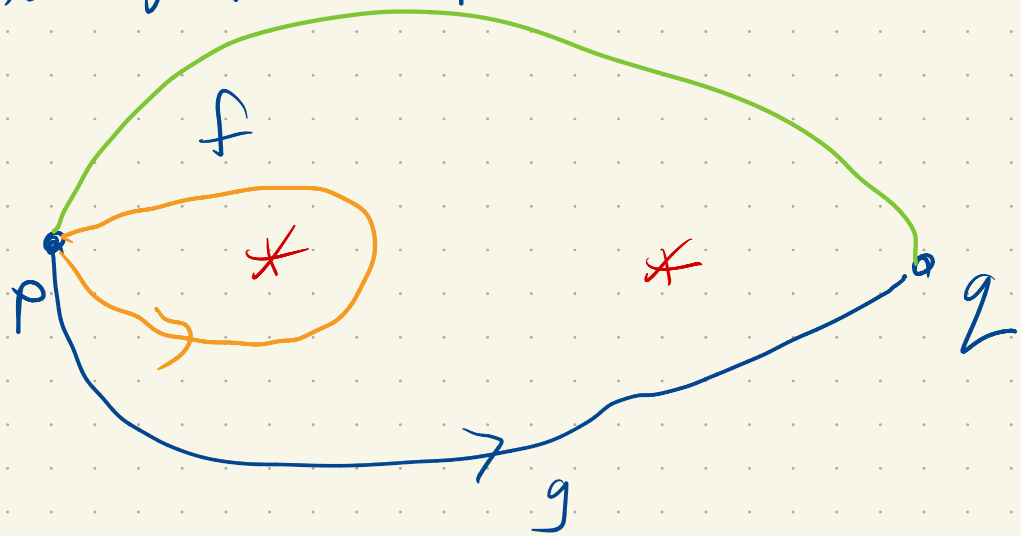
$$= [\bar{g}][f_1][c_p][f_2][g]$$

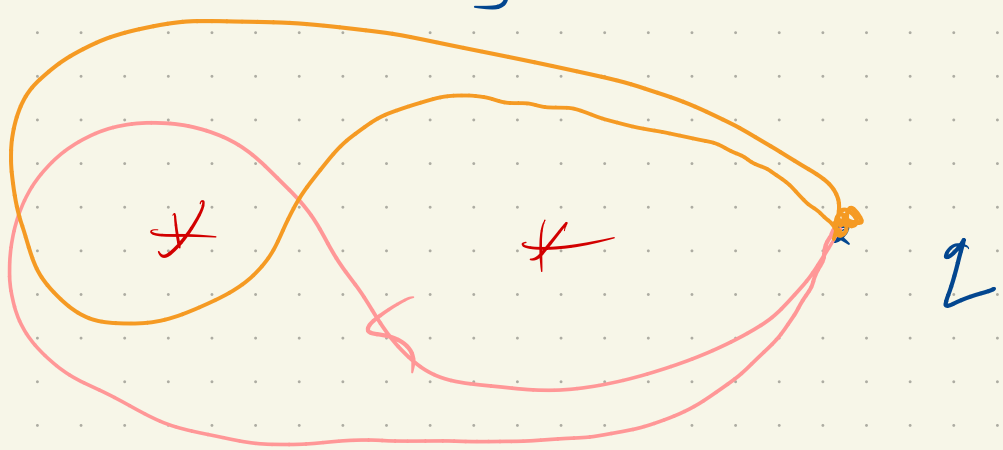
$$= [\bar{g}][f_1][g][\bar{g}][f_2][g]$$

$$= \Phi_g([f_1])\,\Phi_g([f_2]))$$

Exercise: $\Phi_g^{-1} = \Phi_{\bar{g}}$

Warning: the group isomorphism depends on g.





Def: A space is simply connected if it is path connected and at some point its fundamental group is trivial.

(and hence every),

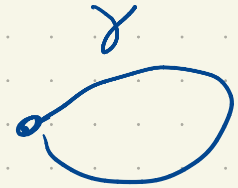
Exercise: Convex subsets of $\mathbb{R}^n$ are simply connected.

$S^1 \longrightarrow \quad \pi_1(S^1) \cong \mathbb{Z}$

$\pi_1(S^n) \cong$ trivial $\quad n \geq 2$

$$X \xrightarrow{f} Y$$

$p \qquad f(p)$



$$f_*([\gamma]) = [f \circ \gamma]$$

$$\pi_1(X, p) \xrightarrow{f_*} \pi_1(Y, f(p))$$

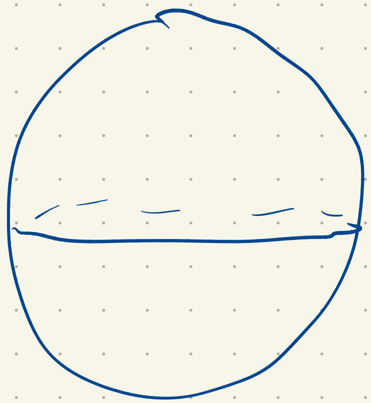
"push forward"

f_* is a group homomorphism

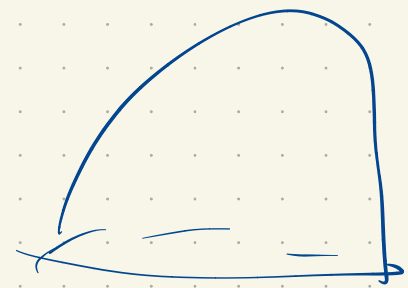
$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g \circ f"] & \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

$$\begin{tikzcd} \pi_1(X, p) \arrow[d, "f_*"'] \arrow[dr, "g_* \circ f_*"] & \\ \pi_1(Y, f(p)) \arrow[r, "g_*"'] & \pi_1(Z, g(f(p))) \end{tikzcd}$$

$\pi_1(S^2)$ is trivial



S^2



$\mathbb{RP}^2$

$\mathbb{Z}_2$

$\mathbb{RP}^1 \sim S^1$