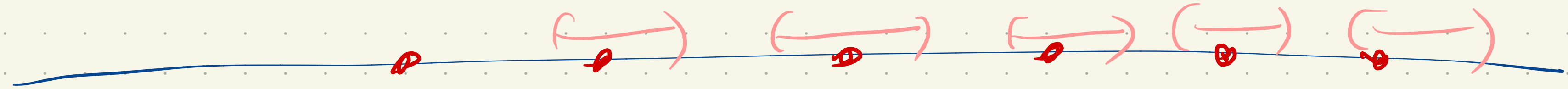
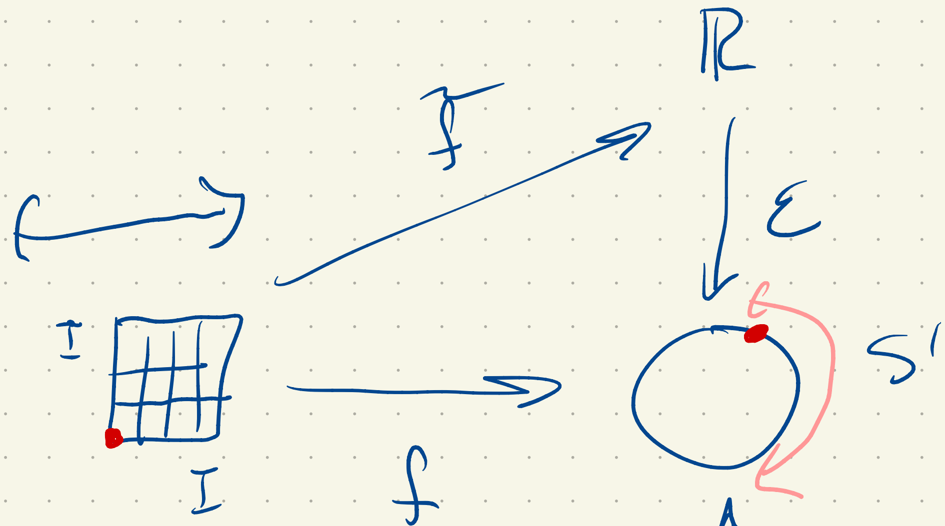
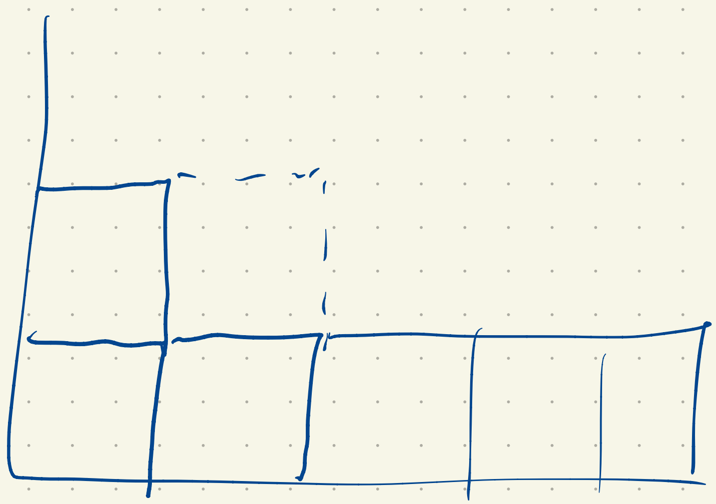
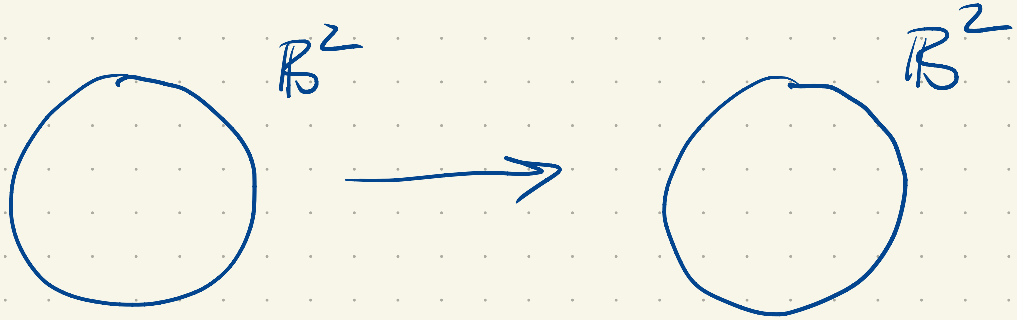




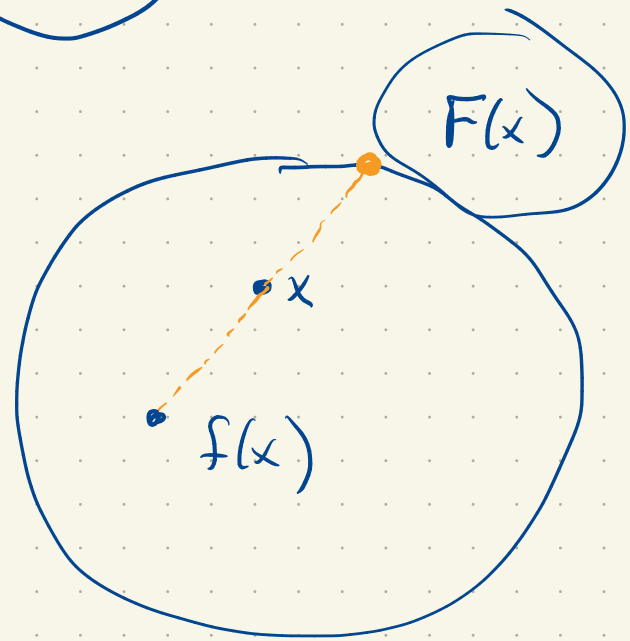
evenly covered nbhds

$\{V_\alpha\}$





$f(x) = x$



$F(x)$ is continuous. $\mathbb{B}^2 \to S^1$

If $x_n \to x$ then $F(x_n) \to F(x)$.

You need to show $\{F(x_n)\}$ converges!

Suppose $x_n \to x$ and $F(x_n) \to z$.

Need to show $z = F(x)$.

Want to show $\underline{G(F) \text{ is}}$ closed,

$$(x_n, F(x_n)) \longrightarrow (x, z)$$

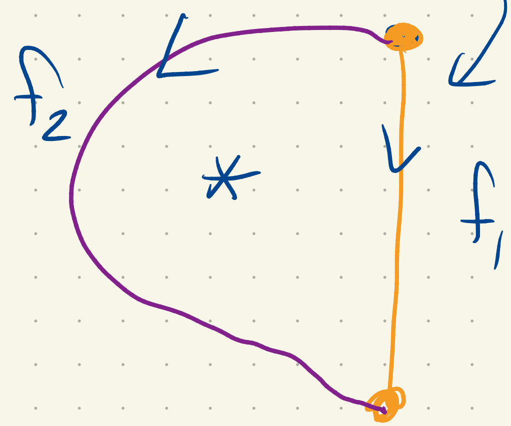
Need to show $(x, z) \in G(F)$

$$\Downarrow$$

$$(x, F(x))$$

Fundamental Group (Topology + Algebra)

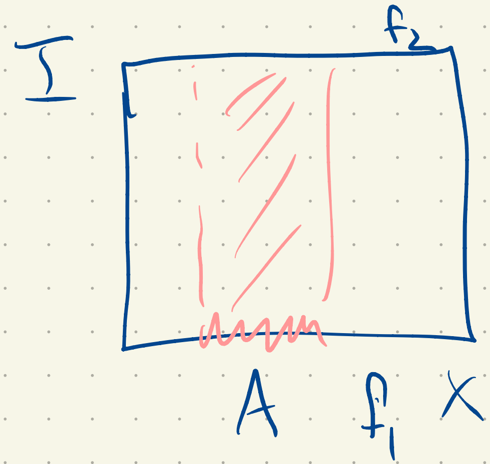
Relative homotopy



both homotopic to constant maps.

Def: Let X, Y be spaces and $A \subseteq X$.
Suppose $f_1, f_2 : X \to Y$. We say they are homotopic relative to A if there is a homotopy H from f_1 to f_2 such that

$$H(a,t) = f_1(a) \quad \text{for all } a \in A.$$



(only possible if

$f_2(a) = f_1(a) \quad \forall a \in A$

Def: Suppose $f_1, f_2 : I \to X$ are two paths,

We say they are <u>path homotopic</u> if

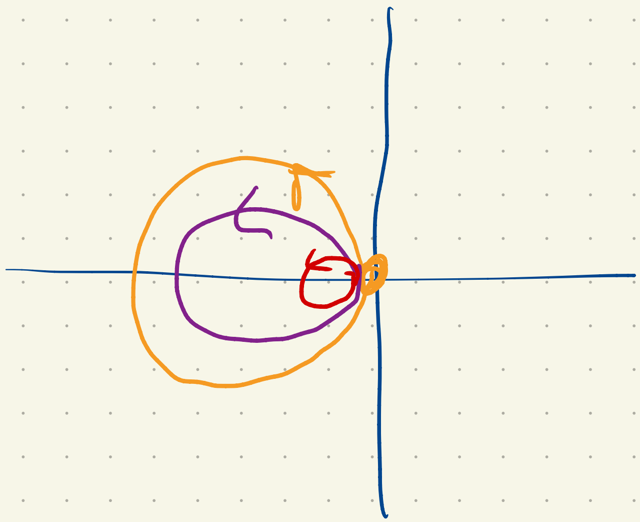
they are homotopic relative to $\{0, 1\}$

(The endpoints need to stay put).

Important special case: A path $f : I \to X$ is

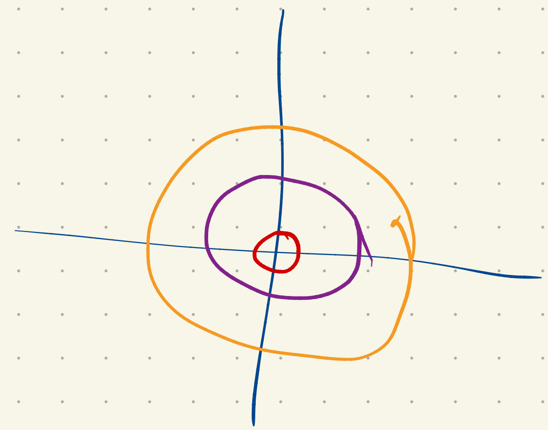
a loop if $f(0) = f(1)$.

e.g. $f(s) = e^{2\pi i s} - 1 \qquad s \in I$



Path homotopy:

$$H(s,t) = f(s)(1-t)$$

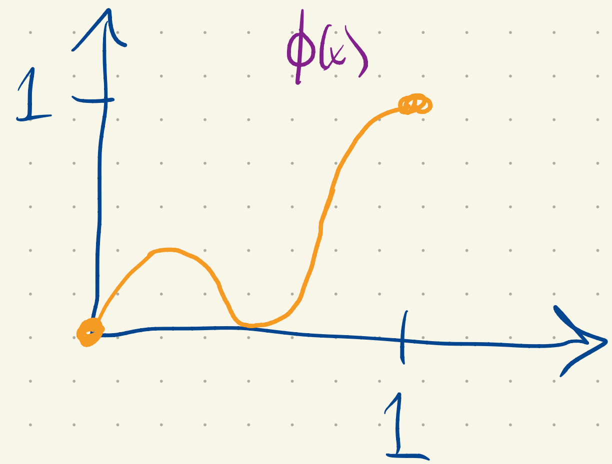


Paths that are path homotopic to constant paths are called null homotopic. (boring)

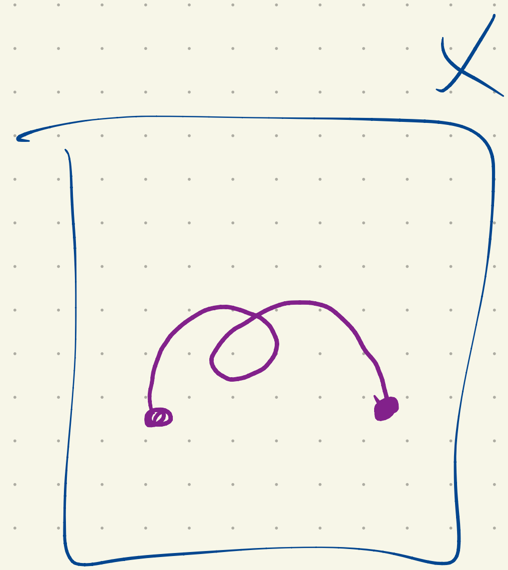
(Must be a loop!)

E.g.: $\phi : I \to I$

$\phi(0) = 0, \; \phi(1) = 1$



Consider a path $f : I \to X$.



We can make a new path $\tilde{f} = f \circ \phi$.

Observe that $\tilde{f}(0) = f(\phi(0)) = f(0)$

and also $\tilde{f}(1) = f(1)$.

We call $\tilde{f}$ a reparameterization of f.

In fact $\tilde{f}$ is path homotopic to f.

$$H(s,t) = f\big(s(1-t) + \phi(s)t\big)$$

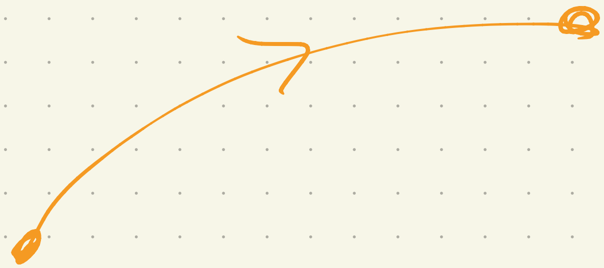
$t=0$ $\quad f(s \cdot 1 + 0) = f(s)$

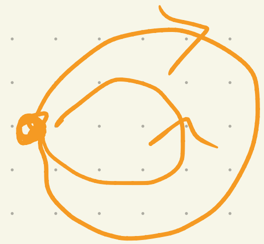
$t=1$ $\quad f(\phi(s)) = \tilde{f}(s)$

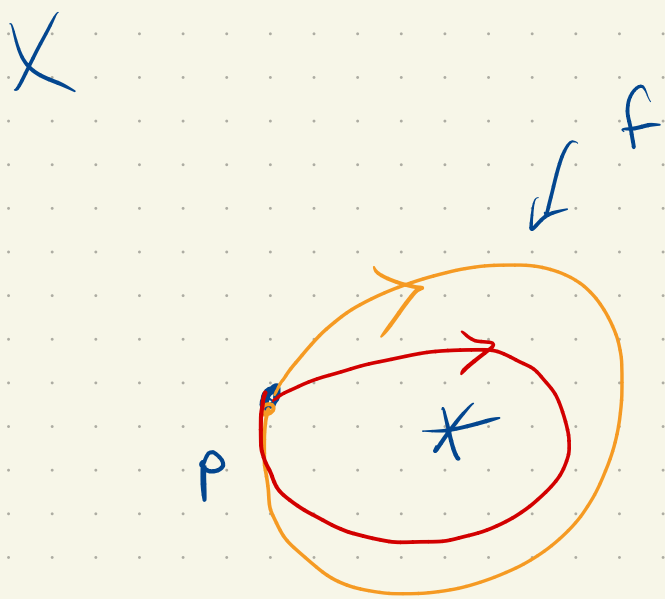
$$H(1,t) = f\big(1(1-t) + \phi(1)t\big)$$

$$= f(1-t+t) = f(1)$$

$H(0,t) = f(0)$ similarly







$[f]$ ← path homotopy class of f

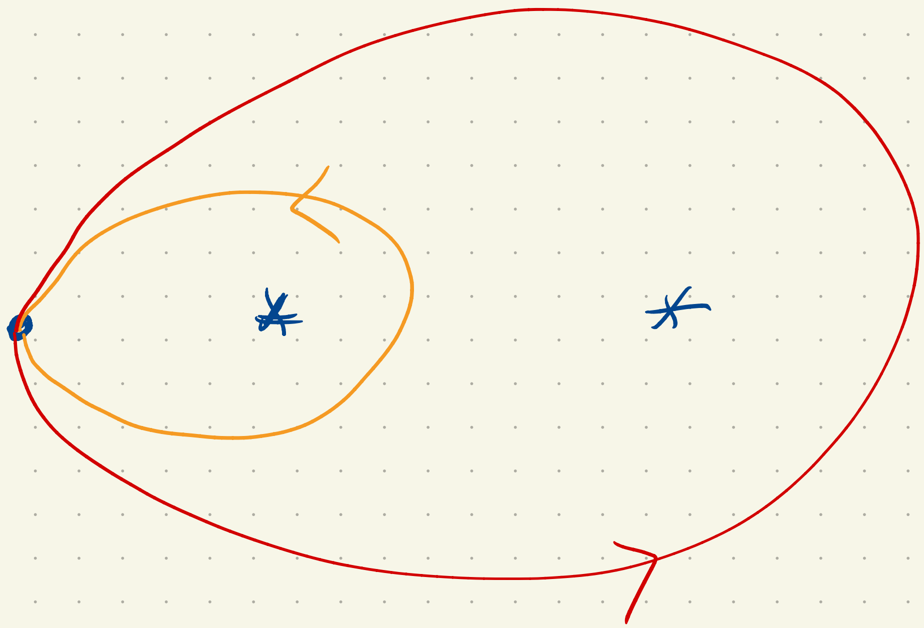
(all paths that are path homotopic to f)

Set of all path homotopy classes of loops based at p

$\pi_1(X, p)$ ← fundamental group of X (with base point p)

$[X, Y]$

$\pi_1(X, p)$ is a group!



Def: Two paths f, g in X are composable

iff $f(1) = g(0)$. (Note: order matters)

If f and g are composable

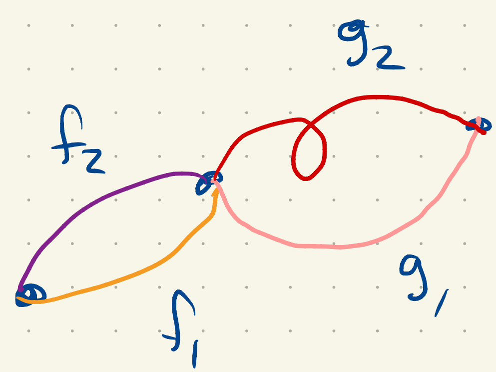
we define

$$f \cdot g\,(s) = \begin{cases} f(2s) & 0 \leq s \leq 1/2 \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Note that $f \cdot g$ is continuous by pasting lemma.

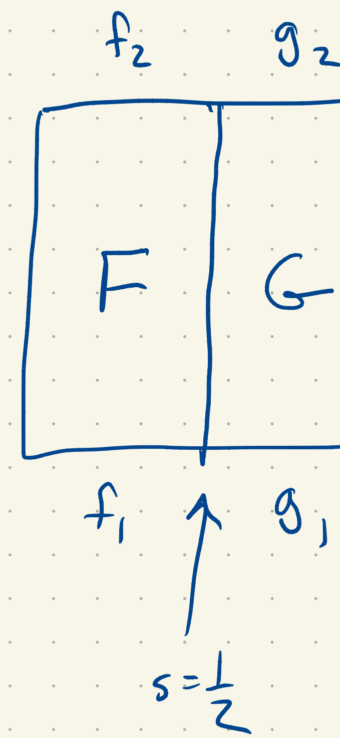
Note: if f and g are loops based at p then they are composable in either order and we can form $f \cdot g$ or $g \cdot f$.

Prop: Suppose f_1, f_2 are path homotopic
g_1, g_2 are path homotopic
and f_1 and g_1 are composable.



Then f_2 and g_2 are composable and $[\, f_2(1) = g_2(0) \,]$

$f_1 \cdot g_1$ is path homotopic to $f_2 \cdot g_2$.



Pf:

$$H(s,t) = \begin{cases} F(2s, t) & 0 \le s \le \frac{1}{2} \\ G(2s-1, t) & \frac{1}{2} \le s \le 1 \end{cases}$$

$$F(2 \cdot \tfrac{1}{2}, t) = F(1, t) = f_1(1)$$

$$G(2 \cdot \tfrac{1}{2} - 1, t) = G(0, t) = g_1(0)$$

$=$

$\Rightarrow$ pasting!

$[f] \cdot [g]$ well defined

$[f \cdot g]$

$$f \cdot g \sim \hat{f} \cdot \hat{g}$$

$$[f \cdot g] = [\hat{f} \cdot \hat{g}]$$