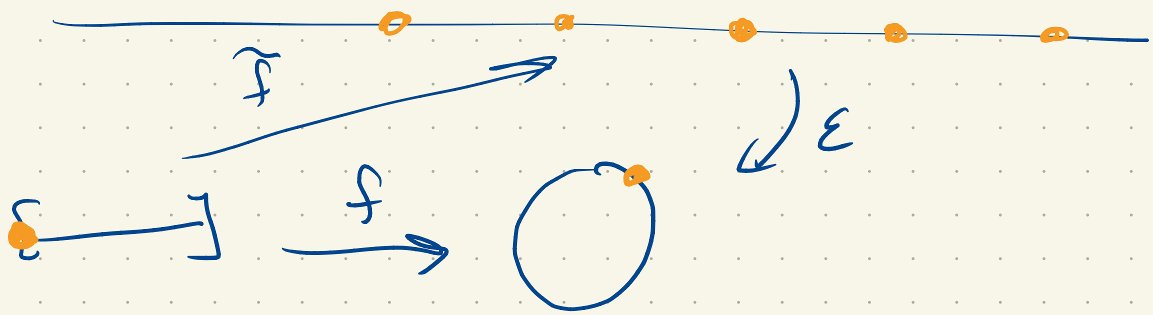
1) Paths into S^1 lift (and you can choose any compatible starting point

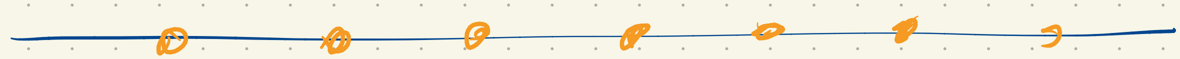


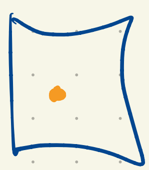
$$\begin{tikzcd}
 & & & \mathbb{R} \arrow[d, "\varepsilon"] \\
I \arrow[urrr, "\tilde{f}"] \arrow[r, "\pi"] \arrow[rrr, bend right, "f"'] & S^1 \arrow[rr, "g"] & & S^1
\end{tikzcd}$$

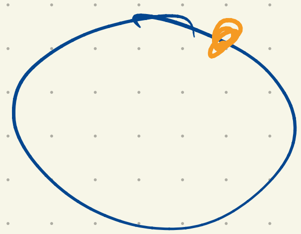
$$\deg(g) = \tilde{f}(1) - \tilde{f}(0)$$

2) Homotopies of paths into S^1 lift (and you can pick one compatible points

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\varepsilon"] \\ I \times I \arrow[ur, "\tilde{F}"] \arrow[r, "F"] & S^1 \end{tikzcd}$$



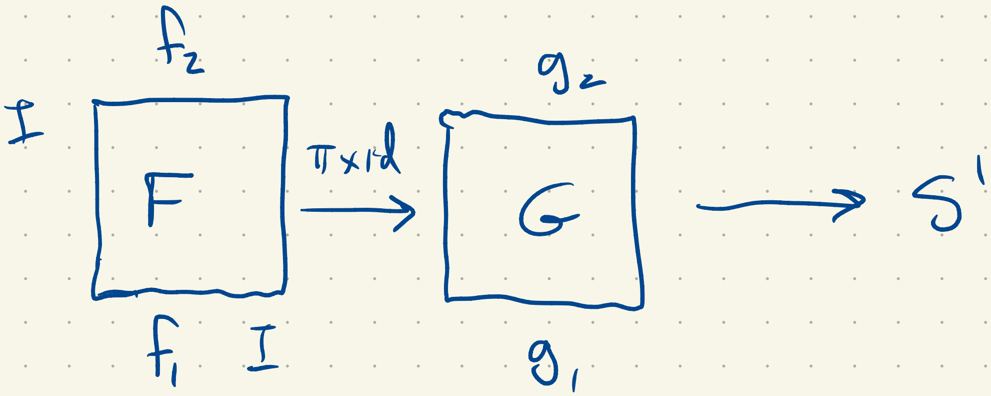




$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ S^1 \arrow[r] & S^1 \end{tikzcd}$$

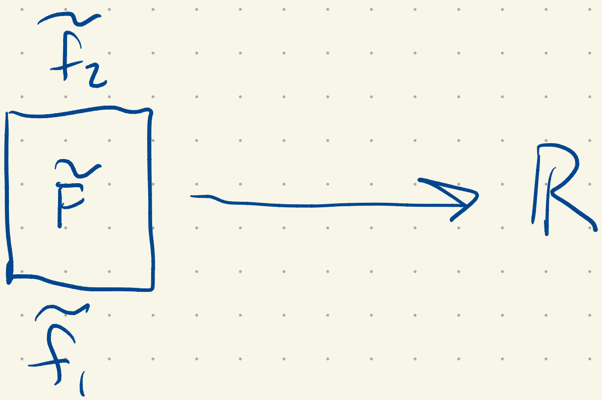
degree descends to homotopy classes

$$S^1 \xrightarrow[g_1,\, g_2]{} S^1 \qquad g_1 \sim g_2 \Rightarrow \deg(g_1) = \deg(g_2)$$



$$\begin{array}{c} F(s,t) \\ \uparrow \\ G(\pi(s),t) \end{array}$$

$$F(s,t) = G(\pi(s),t)$$



$$d(t) = \underbrace{\tilde{F}(1,t) - \tilde{F}(0,t)}_{\mathbb{Z}}$$

$\underline{\deg : [S^1, S^1] \to \mathbb{Z}}$

deg is surjective $\quad \omega_n(z) = z^n \quad \deg(\omega_n) = n$

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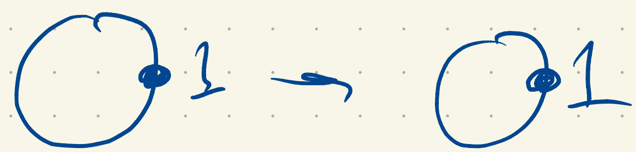
deg is injective

Lemma: $g : S^1 \to S^1$

Then g is homotopic to g' with $g'(1) = 1$



If $\deg(g_1) = \deg(g_2) \implies g_1 \simeq g_2$

$$\deg([g_1]) = \deg([g_2])$$

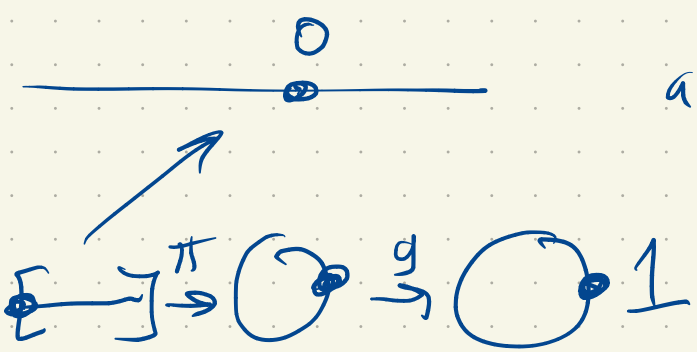
$$\downarrow \qquad\qquad \downarrow$$

$$\deg(g_1) = \deg(g_2) \implies g_1 \simeq g_2$$

$$\implies [g_1] = [g_2]$$

WLOG $g_1(1) = 1$, $g_2(1) = 1$

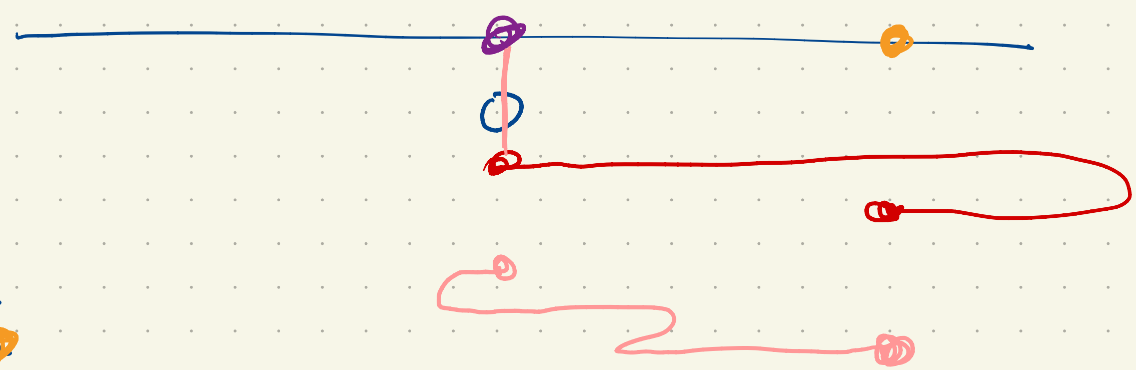
Sketch of proof: Let $f_i = g_i \circ \pi$ and let $\tilde{f}_i$ be a lift of f_i with $\tilde{f}_i(0) = 0$.

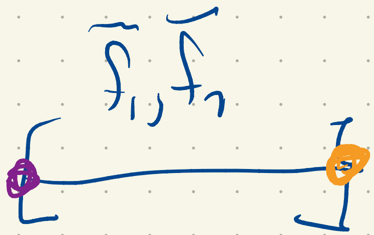


$$\deg(g_1) = \tilde{f}_1(1) - \tilde{f}_1(0) = \tilde{f}_1(1)$$

$$\deg(g_2) = \tilde{f}_2(1)$$

$n = \deg(g_1) = \deg(g_2)$





$$\tilde{H}(s,t) = \tilde{f}_1(s)(1-t) + \tilde{f}_2(s)\,t$$

$$\tilde{H}(0,t) = 0$$

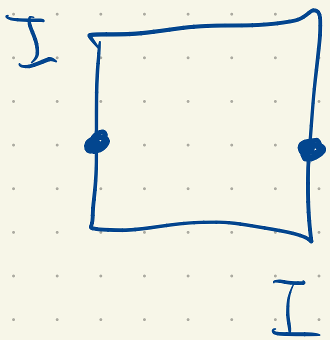
$$\tilde{H}(1,t) = n$$

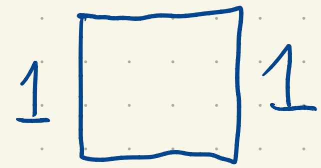
$$H = \varepsilon \circ \tilde{H}$$

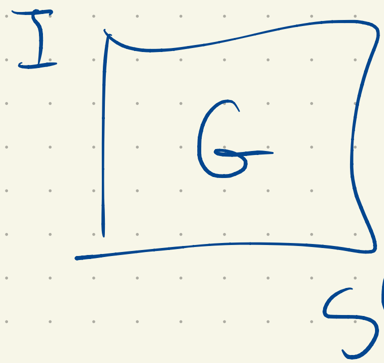
$$\begin{tikzcd}
I \times I \arrow[d, "\pi \circ id"'] \arrow[dr, "H"] & \\
S^1 \times I \arrow[r, "G"'] & S^1
\end{tikzcd}$$

$$H(0,t) = \varepsilon(\tilde{H}(0,t)) = \varepsilon(0) = 1$$

$$H(1,t) = 1$$







$$G(\pi(s), 0) = H(s,0)$$
$$= \varepsilon(\tilde{H}(s,0)) = \varepsilon(\tilde{f}_1(s))$$

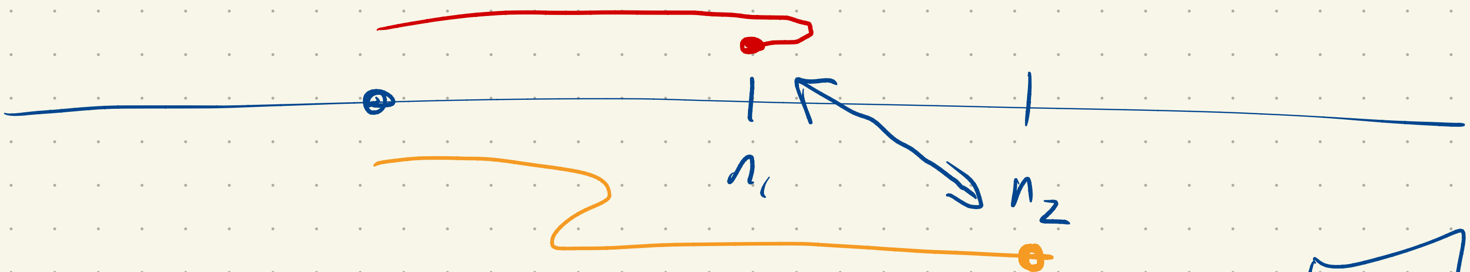
$$= f_1(s)$$

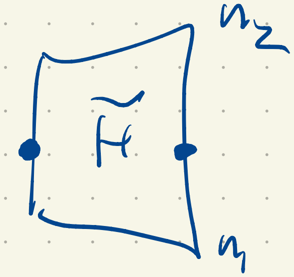
$$\simeq g_1(\pi(s))$$

$$G(z,0) = g_1(z)$$

$$g_1 \simeq g_2$$

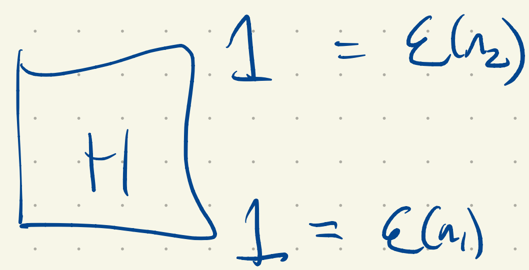
$$G(z,1) = g_2(z)$$





$$\varepsilon \circ \tilde{H}$$

$$\begin{tikzcd} I \times I \arrow[d] \arrow[dr] & \\ S^1 \times I \arrow[r] & S^1 \end{tikzcd}$$



Bower Fixed Point Theorem

Suppose $f: B^2 \to B^2$ is continuous.

Then there exists $x \in B^2$ such that $f(x) = x$.

Why care?

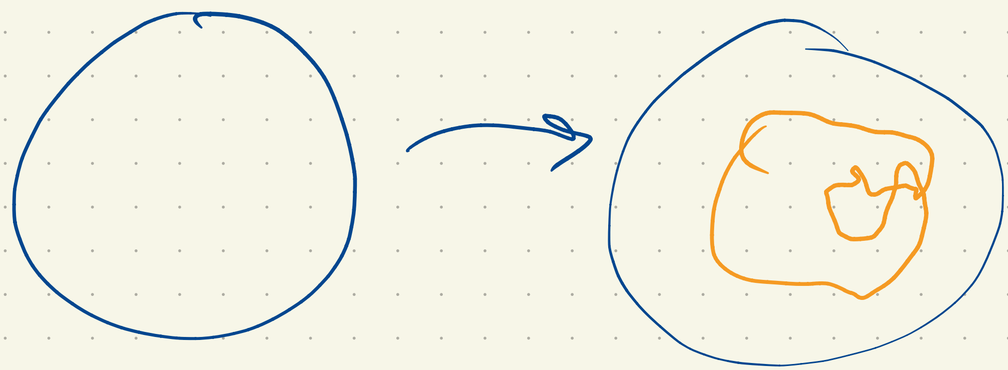
$$F(x) = x^2 - 2$$

$$x_{n+1} = \underbrace{x_n - \frac{F(x_n)}{F'(x_n)}}_{f}$$

$$F(x) = 0$$

$$f(x) = x \iff F(x) = 0$$

$$f(x) = \frac{x}{2} + \frac{2}{x}$$



Lemma: Suppose a_n is a sequence in $\mathbb{R}^2$ and t_n is a sequence in $\mathbb{R}$ and $a_n \to a \neq 0$ and $t_n a_n \to b$.
Then $t_n \to t$ for some t.

$b = 0$ $\quad t_n a_n \to 0$

$a_n \to a \neq 0$

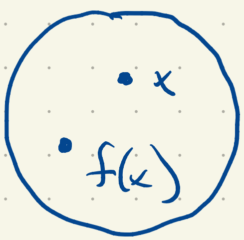
$t_n \to 0$

$b \neq 0$ Then some component is non zero.

Just look at that component to get $t_n \to t$ for some t.

Pf: (of Brouwer)

Suppose to the contrary that $f(x) \neq x$ for all $x \in B^2$.



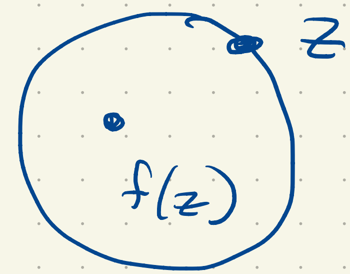
There exists a unique $\overset{\geq 1}{t} = \tau(x)$ such that $f(x) + \underbrace{(x - f(x))}_{\neq 0!}\, t \in S^1 = \partial B^2$.

exercise

We will assume for the moment that τ is continuous.

Define $r(x) = f(x) + (x - f(x))\tau(x)$

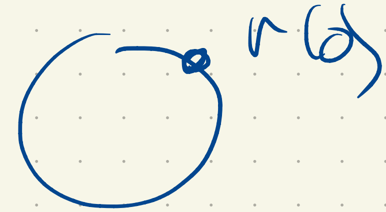
$$r : B^2 \to S^1.$$



This is continuous.

Define $H : S^1 \times I \to S^1$

$$H(z,t) = r(zt)$$



Observe $H(z,0) = r(0)$

which is constant in z.

On the other hand $H(z,1) = z$

since $\tau(x) = 1$ if $x \in S^1$.

Hence H is a homotopy between some constant map

$S^1 \to S^1$ to $S^1 \xrightarrow{id} S^1$.

$\deg(c) = 0$ $\qquad\qquad$ $\deg(id) = 1$

The degrees differ but homotopic maps have the same degree.