$\deg([f])$ $\quad f: S^1 \to S^1$

"count # of wraps"

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\varepsilon"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

$\varepsilon(x) = e^{2\pi i x}$

Lemma: Suppose $\tilde{f}_1$ and $\tilde{f}_2$ are lifts of $f: X \to S^1$ where X is connected. Then there exists $n \in \mathbb{Z}$ such that $\tilde{f}_1(x) = \tilde{f}_2(x) + n$ for all $x \in X$.

Pf: Let $\tilde{f}(x) = \tilde{f}_1(x) - \tilde{f}_2(x)$.

Then, for all $x \in X$,

$\varepsilon(y) = e^{2\pi i y}$

$$\begin{aligned}
\mathcal{E} \circ \tilde{f}(x) &= \mathcal{E}(\tilde{f}_1(x) - \tilde{f}_2(x)) \\
&= e^{2\pi i(\tilde{f}_1(x) - \tilde{f}_2(x))} \\
&= e^{2\pi i \tilde{f}_1(x)} / e^{2\pi i \tilde{f}_2(x)} \\
&= \mathcal{E} \circ \tilde{f}_1(x) / \mathcal{E} \circ \tilde{f}_2(x) \\
&= f(x) / f(x) \\
&= 1.
\end{aligned}$$

So, for all $x \in X$, $\tilde{f}(x) \in \mathcal{E}^{-1}(\{1\}) = \mathbb{Z}$.

Since $\tilde{f}$ is continuous and since $\mathbb{Z}$ is discrete $\tilde{f}$ is constant. Hence there exists $n \in \mathbb{Z}$ such that

$$\tilde{f}_1(x) - \tilde{f}_2(x) = f(x) = n_0$$

Goal: Paths always lift.

Lebesgue Number Lemma: Let X be a compact metric space and let $\{V_\alpha\}$ be an open cover of X. There exists $\varepsilon > 0$ such that for all $x \in X$ there exists α such that $B_\varepsilon(x) \subseteq V_\alpha$.

We call ε a Lebesgue number for the covering.

Pf: Each $x \in X$ is contained in an open V_{α_x} and hence there exists ε_x such that $B_{2\varepsilon_x}(x) \subseteq V_{\alpha_x}$.

The balls $B_{\varepsilon_x}(x)$ cover all X which is compact and we can find $x_1, \dots, x_n$ and radii $\varepsilon_i := \varepsilon_{x_i}$

such that $B_{\varepsilon_i}(x_i)$ cover X.

Let $\varepsilon = \min(\varepsilon_1, \dots, \varepsilon_n)$. I claim ε is

a Lebosgue number.

Let $y \in X$. Then there exists i with

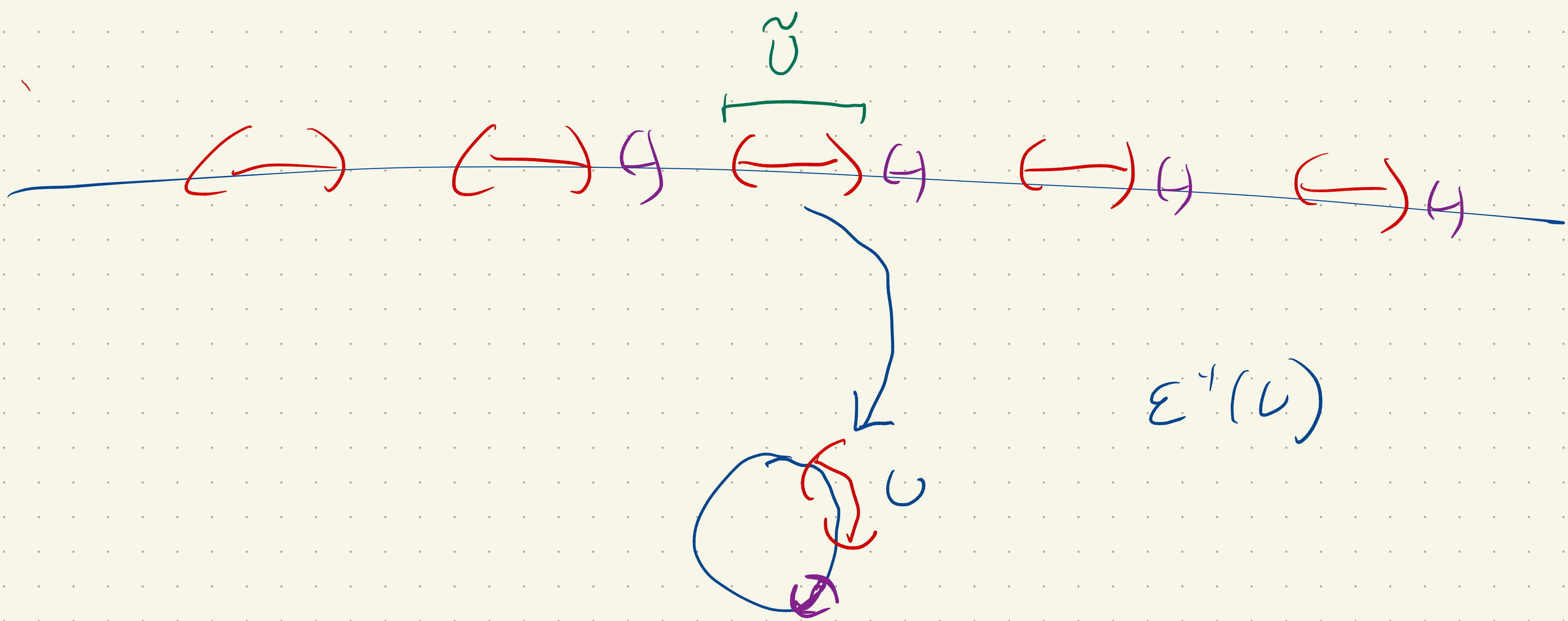
$y \in B_{\varepsilon_i}(x_i)$. Consider some $z \in B_\varepsilon(y)$.

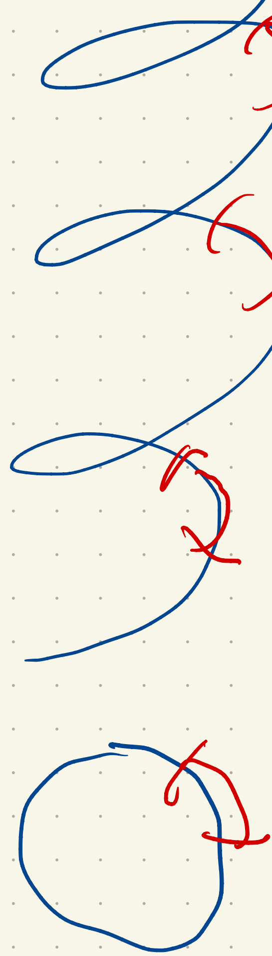
Then
$$
\begin{aligned}
d(z, x_i) &\leq d(z, y) + d(y, x_i) \\
&< \varepsilon + \varepsilon_i \\
&\leq 2\varepsilon_i .
\end{aligned}
$$

Hence $B_\varepsilon(y) \subseteq B_{2\varepsilon_i}(x_i) \subseteq V_{\alpha_{x_i}}$. $\square$

Def: An open set $U \subseteq S^1$ is <u>evenly covered</u> if each component $\tilde{U}$ of $\varepsilon^{-1}(U)$ satisfies

$$\varepsilon\big|_{\tilde{U}} : \tilde{U} \to U \quad \text{is a homeomorphism}$$

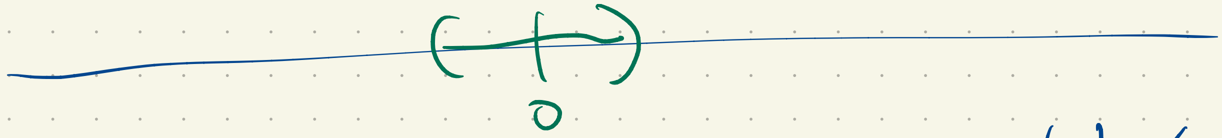




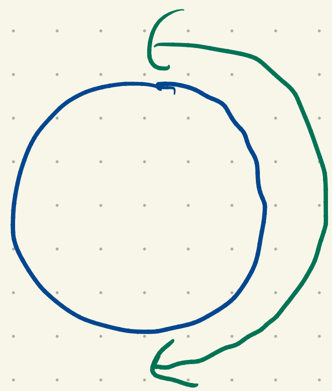
"stack of pancakes"

Prop: Every $z \in S^1$ is contained in an evenly covered neighborhood.

See prop 8.1 of Lee.



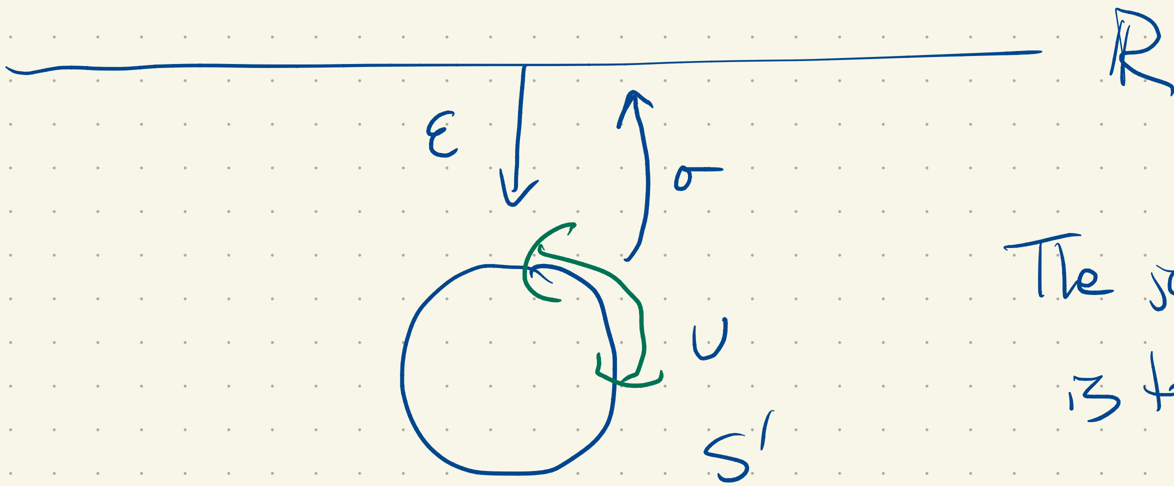
$$\bigcup_{k\in\mathbb{Z}} \left(k-\tfrac{1}{4}, k+\tfrac{1}{4}\right)$$



$x > 0$

Def: Let $U \subseteq S^1$.

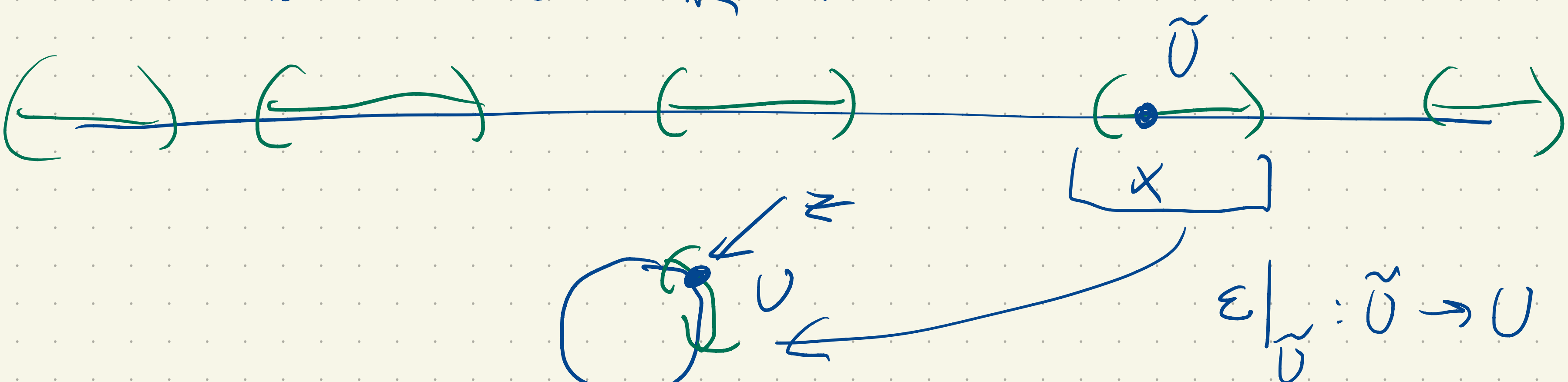
A local section of U is a continuous map $\sigma : U \to \mathbb{R}$ such that $\varepsilon \circ \sigma = id$



consistently

The job of σ is to locally, continuously, assign angles to points in U.

Prop: Suppose $U \subseteq S^1$ is evenly covered and $z \in U$ and $x \in \varepsilon^{-1}(\{z\})$. Then there exists a local section $\sigma : U \to \mathbb{R}$ such that $\sigma(z) = x$.



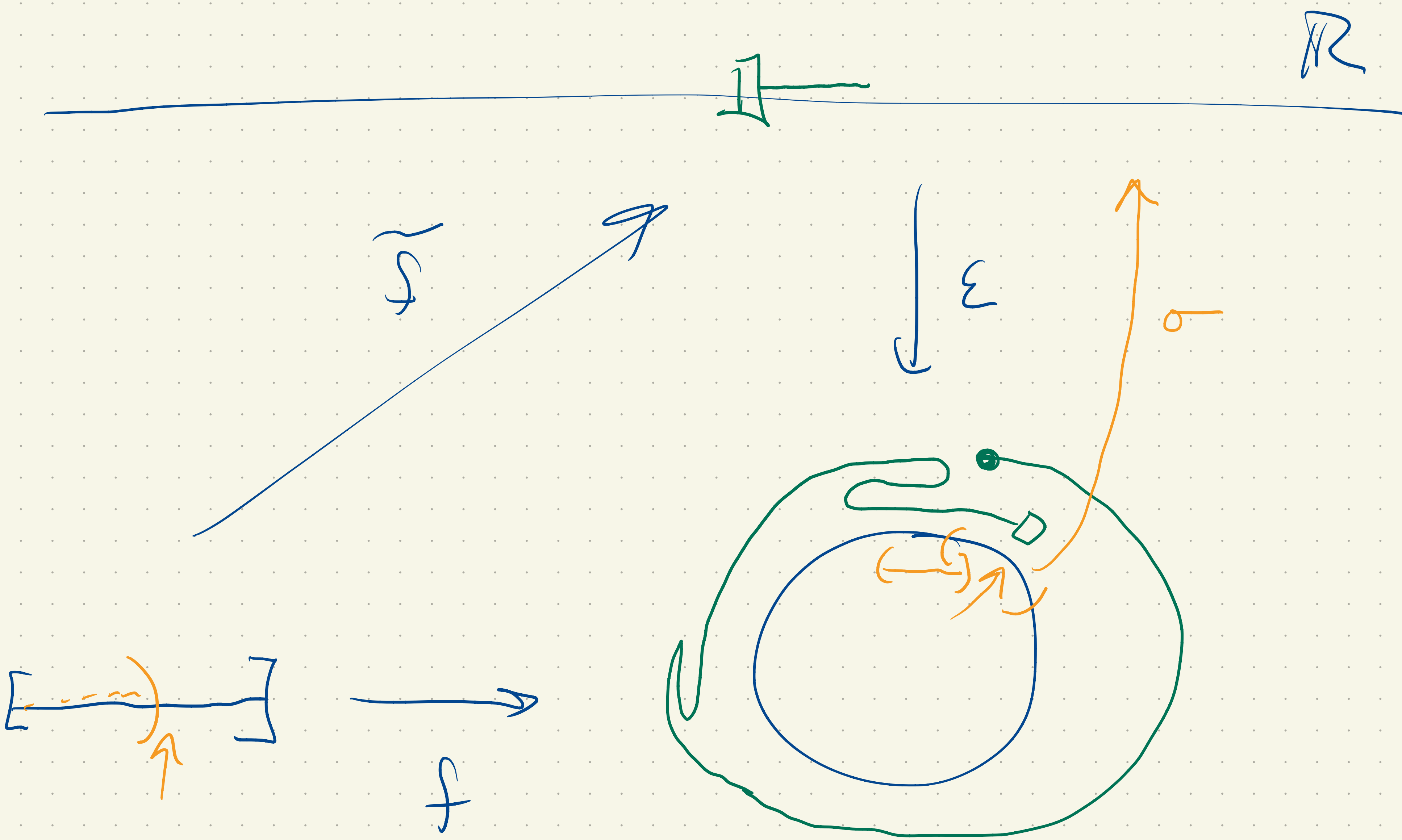
$\varepsilon|_{\tilde{U}} : \tilde{U} \to U$

$$\sigma = \left(\varepsilon\big|_{\tilde{U}}\right)^{-1}$$

Sketch: $\sigma = \left(\varepsilon\big|_{\tilde{U}}\right)^{-1}$.

Real Work

Thm: Suppose $f : I \to S^1$ is continuous $(I = [0,1])$. Then f admits a lift.



$\varepsilon \circ \tilde{f} = f$ $\quad$ $\tilde{f} = \sigma \circ f$