Prop: If X is connected and Y is disconnected then X and Y are not homotopy equivalent.

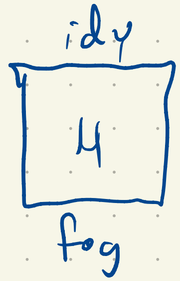
Pf: Let A, B be a separation of Y.

Define $z: Y \to S^0$ by

$$z(y) = \begin{cases} 1 & \text{if } y \in A \\ -1 & \text{if } y \in B \end{cases}$$

Note that z is continuous by the gluing lemma.

Suppose to the contrary that $f: X \to Y$ and $g: Y \to X$ are a homotopy equivalence and hence $f \circ g$ is homotope to id_Y via a homotopy H.



We can assume WLOG $f(X) \subseteq A$.

Pick $p \in B$.

Observe that $z \circ H(p, \cdot): [0,1] \to S^0$

is continuous. Since $[0,1]$ is connected and since S^0 is discrete the map is constant. Observe

$$z(H(p,0)) = z(f(g(p))) = 1 \text{ . since } f(X) \subseteq A_0$$

but

$$z(H(p,1)) = z(id_Y(p)) = z(p) = -1 \text{ since } p \in B_0$$

This contradicts the fact that the map $t \mapsto z(H(p,t))$ is constant.

Cor: S^0 is not contractible.

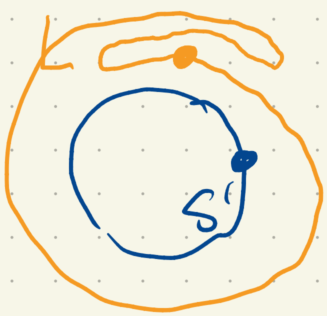
Is S^1 contractible? No, but our tools thus far are not sufficient to show this.

Goal: investigate $[S^1, S^1]$

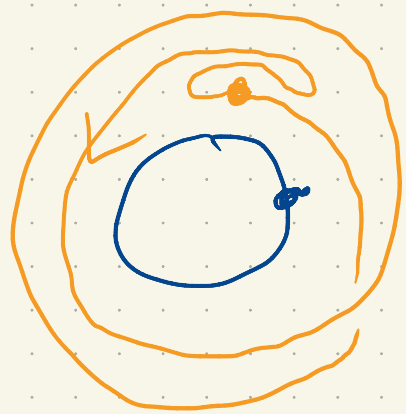
We're going to find a map $\deg : [S^1, S^1] \to \mathbb{Z}$ that is a bijection.

↳ counts the number of times, with orientation, that S^1 wraps around itself.

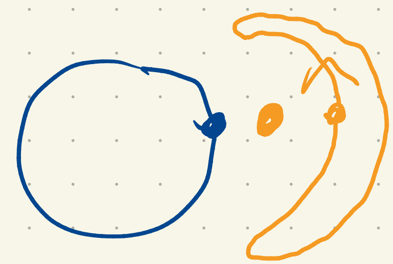
$\deg(f) = 1$



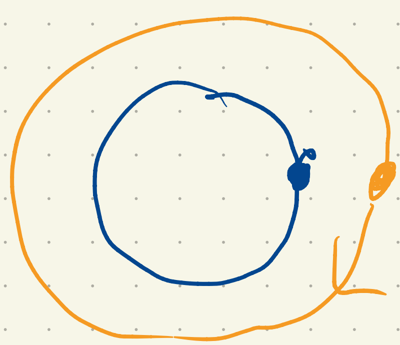
$\deg(f) = 2$



$\deg(f) = 0$



$\deg(f) = -1$



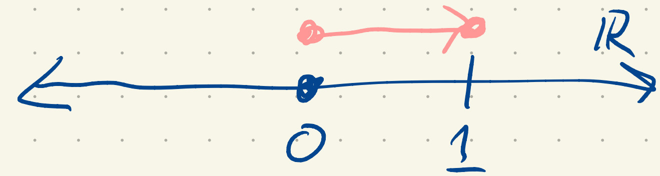
$$\deg : C(S^1, S^1) \to \mathbb{Z}$$

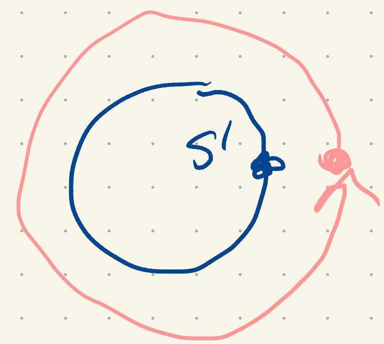
Key tool

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\varepsilon"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

space

$\varepsilon(x) = e^{2\pi i x}$

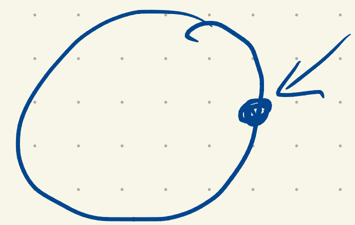




Def: Suppose $f: X \to S^1$ is a map,

A lift of f is a map $\tilde{f}: X \to \mathbb{R}$ such that $\varepsilon \circ \tilde{f} = f$.

$$
\begin{tikzcd}
 & \mathbb{R} \arrow[d, "\varepsilon"] \\
S^1 \arrow[ur, "\widetilde{id}"] \arrow[r, "id"'] & S^1
\end{tikzcd}
$$



If $\widetilde{id}$ existed

$$
\begin{aligned}
[id] &= [\varepsilon \circ \widetilde{id}] \\
&= [\varepsilon] \circ [\widetilde{id}] \\
&= [\varepsilon] \circ [c_0] \\
&= [\varepsilon \circ c_0] \\
&= [c_1]
\end{aligned}
$$

Now show that

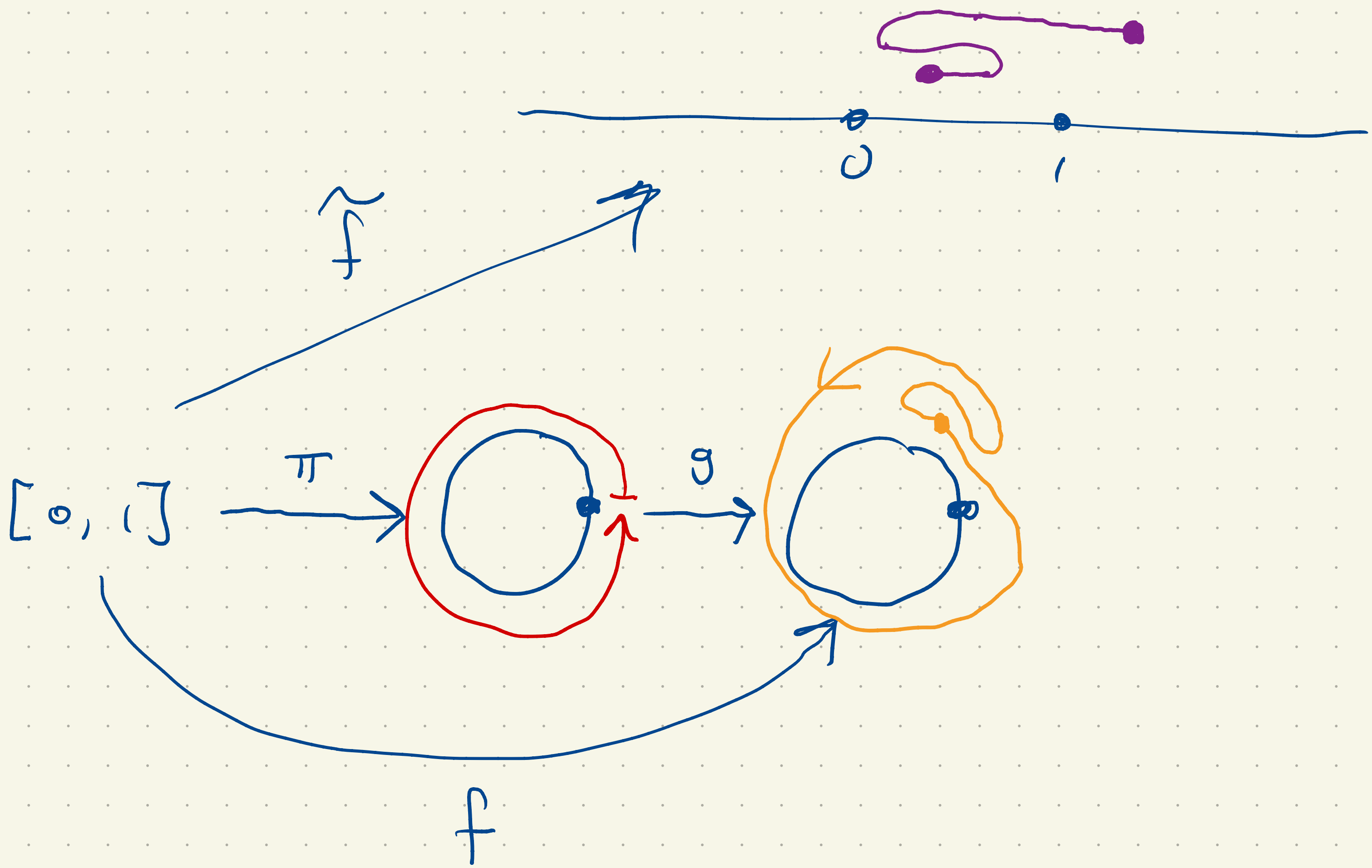
$S^1 \xrightarrow{c_1} \{1\}$

$\{1\} \hookrightarrow S^1$

are homotopy inverses,

We will show that paths into S^1 always lift

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ {[0,1]} \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$



$$\deg(g) = \underbrace{\tilde{f}(1) - \tilde{f}(0)}_{\in \mathbb{Z}}$$

Plan: a) Two lifts of a function on a connected space differ by an integer offset

b) paths lift!

and hence def of $\deg(g)$ is well-defined.

c) If $g_1 \sim g_2$ then $\deg(g_1) = \deg(g_2)$

so $\deg : [S^1, S^1] \to \mathbb{Z}$.

d) If $\deg([g_1]) = \deg([g_2]) \Rightarrow [g_1] = [g_2]$

and here deg is injective

e) $\forall n \in \mathbb{Z}$ there exists $\omega_n : S^1 \to S^1$

with $\deg([\omega_n]) = n$.

In fact $\omega_n(z) = z^n$ will work.