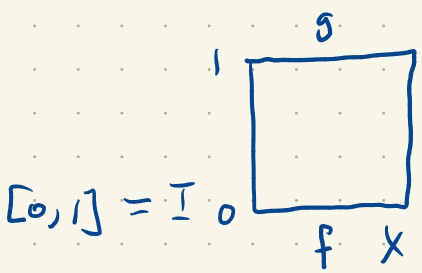
$X, Y \qquad f, g : X \to Y$

homotopy from f to g



$H : X \times I \to Y$

cts

$H(x, 0) = f(x)$

$H(x, 1) = g(x)$

$f \sim g$ "f is homotopic to g"

$\sim$ is an equivalence relation.

$[f]$ ← homotopy class of f

all functions homotopic to f

$C(X, Y)$ all cts functions from X to Y

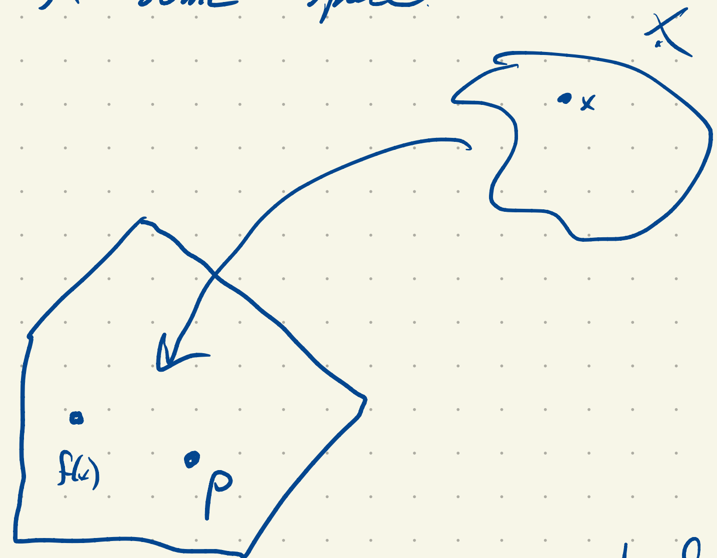
$[X, Y]$ ← set of homotopy classes of functions from X to Y.

E.g. $K \subseteq \mathbb{R}^n$, convex, $p \in K$, X some space

$$[X, K] = \{ [c_p] \}$$

$c_p : X \to K$

$c_p(x) = p$



$\overbrace{p(1-t) + qt}^{\text{convex comb of } p,\, q} \in K$

for all $t \in [0,1]$

$f : X \to K$

$H(x,t) = f(x)\cdot(1-t) + pt$

$(x,t) \longmapsto f(x)$

$(\underset{\mathbb{R}^n}{z}, \underset{\mathbb{R}}{t}) \to zt$

$(x,t) \longmapsto (\underset{K}{f(x)}, \underset{\mathbb{R}}{(1-t)}) \to f(x)(1-t)$

$(x,t) \longmapsto (p,t) \to (pt)$

$$(x,t) \longmapsto \big(f(x)(1-t),\ pt \big) \longrightarrow f(x)(1-t) + pt$$

$$f : X \to X$$

$$f \sim c_p \qquad [f] = [c_p]$$

$$[X, K] = \{ [c_p] \}$$

Suppose $X \underset{f_2}{\overset{f_1}{\longrightarrow}} Y \underset{g_2}{\overset{g_1}{\longrightarrow}} Z$

$$f_1 \sim f_2 \quad , \quad g_1 \sim g_2$$

Is $g_1 \circ f_1 \sim g_2 \circ f_2$

$$f_1 \overset{F}{\sim} f_2$$

$$g_1 \overset{G}{\sim} g_2$$

$$H(x,t) = G(F(x,t),t)$$

$$\begin{aligned} H(x,0) &= G(F(x,0),0) \\ &= G(f_1(x),0) \\ &= g_1(f_1(x)) \end{aligned}$$

$(x,t) \longmapsto F(x,t)$

$(y,s) \longmapsto G(y,s)$

$(x,t) \longmapsto (F(x,t),t)$

$$H(x,1) = G(F(x,1),1)$$
$$= G(f_2(x),1)$$
$$= g_2(f_2(x))$$

$X \to Y \to Z$

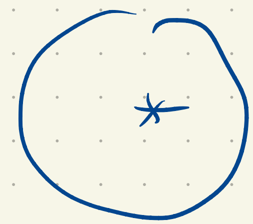
$[f] \qquad [g]$

$$[g]\circ[f] \in [X,Z]$$

$$\begin{matrix} \uparrow & & \uparrow \\ \hat{g} & & \hat{f} \end{matrix} \qquad [\hat{g}\circ\hat{f}]$$

When are two spaces the same from the perspective of homotopy?

$\mathbb{R}^2 \setminus \{0\}$ $\qquad S^1$



$$f : \mathbb{R}^2 \setminus \{0\} \to S^1$$

$$f(x) = \frac{x}{\|x\|}$$

$$g : S^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}$$

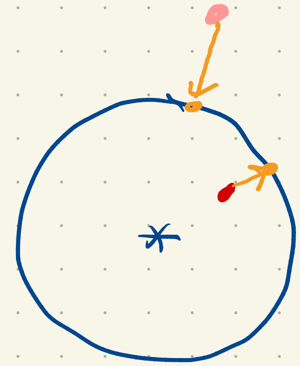
$$g(x) = x$$

$$(f \circ g)(x) = f(x) = \frac{x}{\|x\|} = x$$

$$f \circ g = id_{S^1}$$

$$g(f(x)) = g\left(\frac{x}{\|x\|}\right) = \frac{x}{\|x\|}$$

$$g \circ f \sim id_{\mathbb{R}^2 \setminus \{0\}}$$



$$H(x,t) = x(1-t) + \frac{x}{\|x\|} t$$

$$\hookrightarrow X \times I \longrightarrow X$$

$\uparrow$

$\mathbb{R}^2 \setminus \{0\}$

$H(x,t) = 0$?

$$x(1-t) + \frac{x}{\|x\|} t = x\left[(1-t) + \frac{1}{\|x\|} t\right]$$

$$g \circ f \sim id_X$$

$$f \circ g = id_Y \sim id_Y \qquad Y = S^1$$

Def: Two spaces X, Y are <u>homotopy equivalent</u> if

there are (continuous) maps $f: X \to Y$
$g: Y \to X$

such that $g \circ f \sim id_X$
$f \circ g \sim id_Y$.

We call the maps f, g <u>homotopy equivalences</u>.

We just saw that $\mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to S^1!

e.g. $X = \mathbb{R}^n$, $Y = \{p\}$ I claim X and Y are homotopy equivalent.

$$\mathbb{R}^n \xrightarrow{c_p} Y \qquad c_p \circ c_0 = id_Y \sim id_Y$$

$$Y \xrightarrow{c_0} \mathbb{R}^n \qquad c_0 \circ c_p(x) = c_0(p) = 0.$$

$c_0 \circ \varepsilon_p \sim \mathrm{id}_{\mathbb{R}^n}$? Yes by the earlier convexity argument.

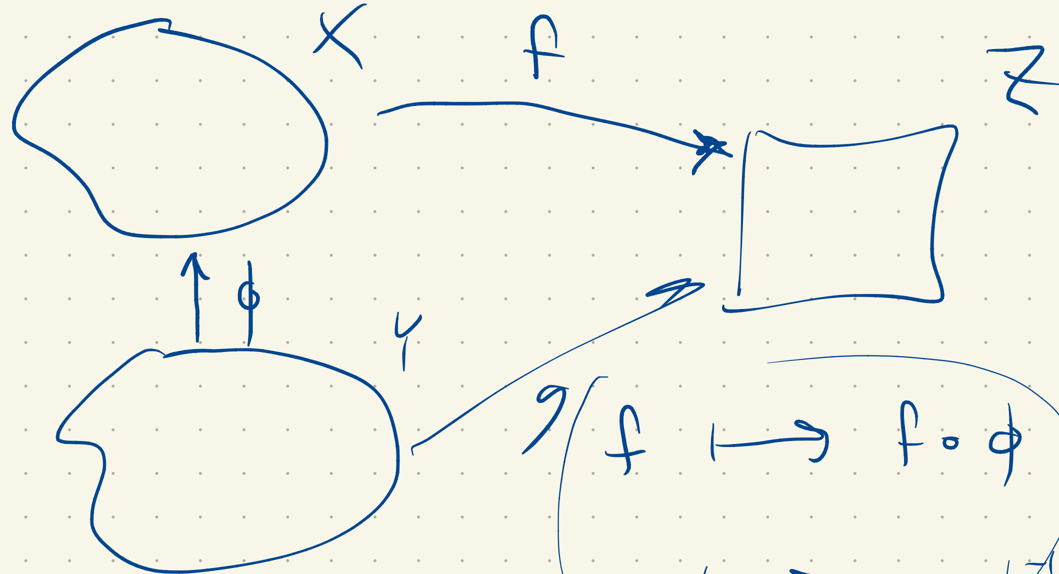
$\mathbb{R}^n$ (any convex subset of $\mathbb{R}^m$!) is homotopy equivalent to a 1-point space.

Def: A space is <u>contractible</u> if it is homotopy equivalent to a 1 point space.

Suppose $f: X \to Y$, $g: Y \to X$ are homotopy equivalences.

Let Z be a space.

$$\begin{tikzcd}
{[X, Z]} \arrow[d, bend left, leftrightarrow] \\
{[Y, Z]}
\end{tikzcd}$$



$f \longmapsto f \circ \phi$

$g \longmapsto g \circ \phi^{-1}$

$f \circ \phi \circ \phi^{-1} = f$

$f : X \to Y$

$X, Y \qquad f, g \qquad Z \qquad g : Y \to X$

$[\ell] \circ [f] \qquad [X, Z] \quad \cdots \quad [k] \qquad k : X \to Z$

$\uparrow \qquad \downarrow$

$[\ell] \qquad [Y, Z] \qquad [k] \circ [g]$

$$[k] \to [k] \circ [g] \to \underbrace{([k] \circ [g]) \circ [f]}$$

$$[k \circ g] \circ [f]$$

$$[k \circ g \circ f]$$

$$[k] \circ [g \circ f]$$

$$[k] \circ [id_x]$$

$$[k \circ id_x]$$

$$[k]$$

X, Y

$X \xrightarrow{f} Y$

$Y \xrightarrow[g]{} X$

$g \circ f \sim id_X$

$f \circ g \sim id_Y$

$X \longrightarrow Z$

$Y \longrightarrow Z$

$$\begin{array}{ccc} [X, Z] & & [Y, Z] \\ \uparrow & & \\ [h] & & [h] \circ [g] \end{array}$$

$[Z, X] \longleftrightarrow [Z, Y]$