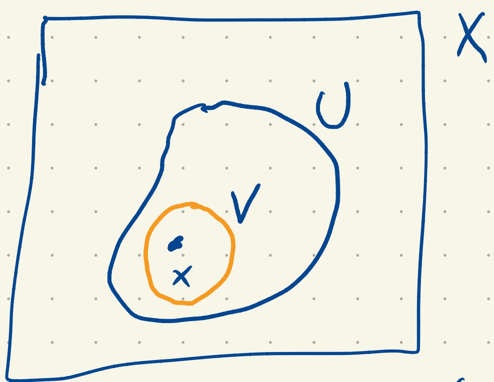
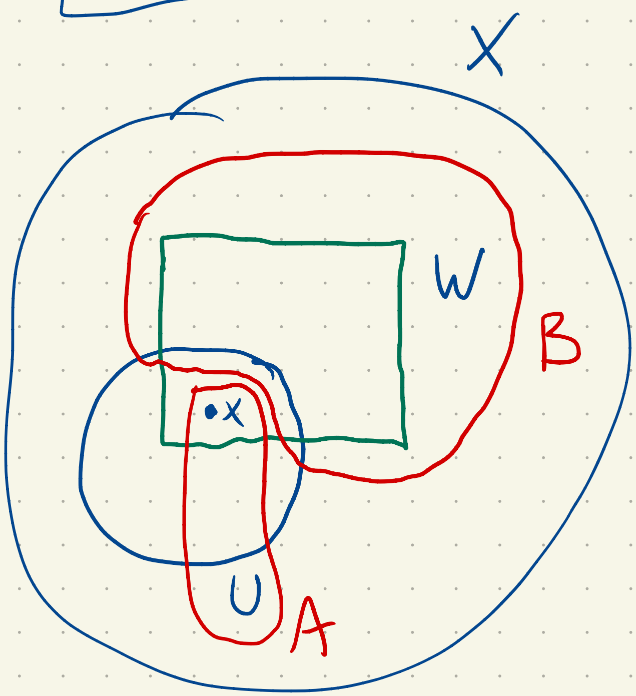
Shrinking Lemma If X is a locally compact Hausdorff space and $U \subseteq X$ is open and $x \in U$, there exists an precompact open set V such that $x \in V \subseteq \overline{V} \subseteq U$.





Pf: Let x and U be given as in the statement. Since X is LCH - there is a precompact open set W containing x. Observe $\overline{W} \setminus U$ is a closed subset of $\overline{W}$ and is hence compact. Since X is Hausdorff and since $x \notin \overline{W} \setminus U$, there exist disjoint open sets A and B such that $x \in A$ and $\overline{W} \setminus U \subseteq B$.

Let $V = A \cap W$ which is an open set that contains x.

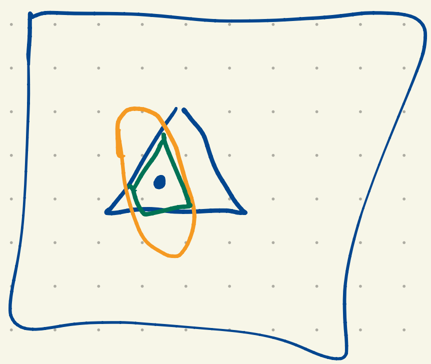
Observe that $\overline{V} \subseteq \overline{W}$ since $V \subseteq W$.

Moreover, since $V \subseteq A \subseteq B^c$ it follows that $\overline{V} \subseteq B^c \subseteq (\overline{W} \setminus U)^c$.

closed!

$$
\begin{aligned}
\text{Hence } \overline{V} &\subseteq \overline{W} \cap (\overline{W} \setminus U)^c \\
&= \overline{W} \cap (\overline{W} \cap U^c)^c \\
&= \overline{W} \cap (\overline{W}^c \cup U) \\
&= (\overline{W} \cap \overline{W}^c) \cup (\overline{W} \cap U) \\
&= \overline{W} \cap U \subseteq U.
\end{aligned}
$$

Lemma: Every closed subset of a LCH is LCH.



Prop: An open subset of a LCH is LCH.

Pf: Let U be open in the LCH space X.

Suppose $x \in U$. From the shrinking lemma we can find a precompact open set $V \subseteq U$ such that $cl(V, X) \subseteq U$.

Now $cl(V, U) = cl(V, X) \cap U = cl(V, X)$.

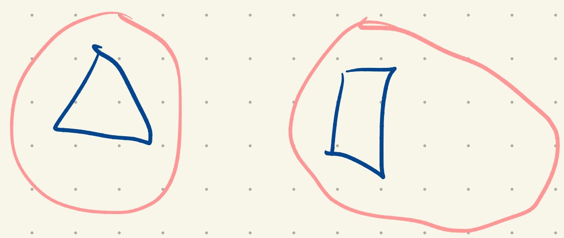
Moreover $cl(V, X)$ is compact with respect to X and hence also with respect to U.

Cor: Every open subset of a compact Hausdorff space is LCH.

In fact, given a LCH X there is a compact Hausdorff space X^* such that $X \subseteq X^*$ (with the subspace top) and $X^* \setminus X$ has just one point. (one-point compactification).

Important omissions

- $\overset{T^4}{\text{normal}}$ and $\overset{T^3}{\text{regular}}$ spaces
  generalizations of Hausdorffness

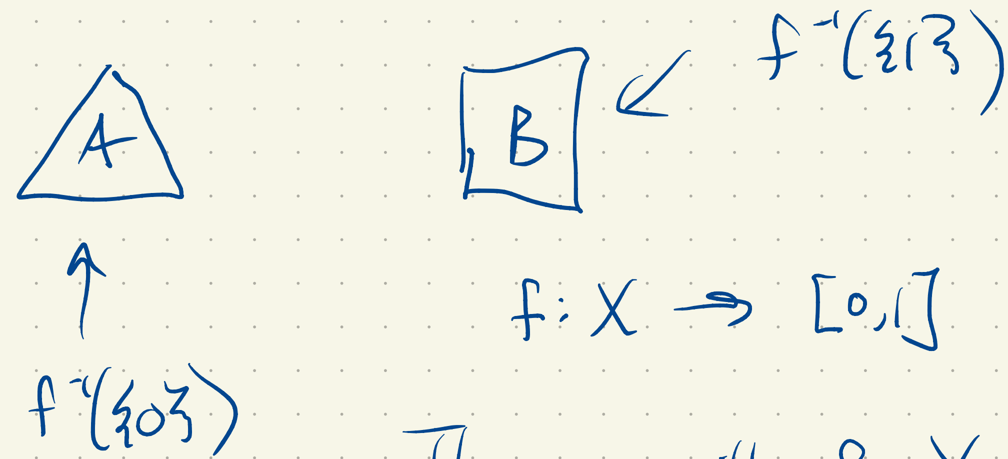


- Urysohn metrization theorem
  (gives decent sufficient conditions for a

top space to be metrizable)

(2nd countable + regular) $\Rightarrow$ (2nd countable + normal)

- Urysohn Lemma



$f: X \to [0,1]$

This is possible if X is normal.

- Tycohoff's Theorem (an arbitrary product of compact spaces is compact).

Homotopy

$\mathbb{R}^1 \not\cong \mathbb{R}^n$ for $n \neq 1$ via a connectivity argument

$$X \xrightarrow[f,\,g]{} Y$$

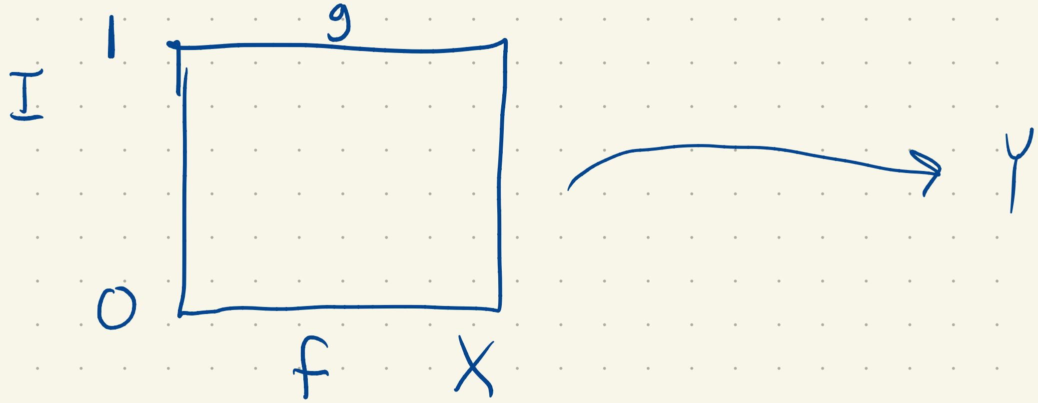
We say f, g are <u>homotopic</u> if there exists

a continuous map $H : X \times I \to Y$ $\quad (I = [0,1])$

such that
$$\left.\begin{aligned} H(x,0) &= f(x) \text{ for all } x \in X \\ H(x,1) &= g(x) \text{ for all } x \in X. \end{aligned}\right]$$

we call H a <u>homotopy</u> from f to g

"f can be continuously deformed into g"



e.g. $X = Y = \mathbb{R}$

$H(x,t) = x(1-t)$

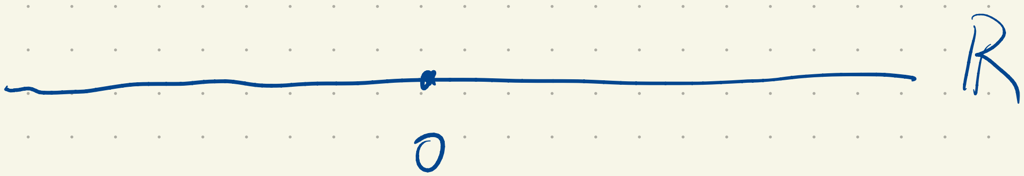
$f(x) = x$

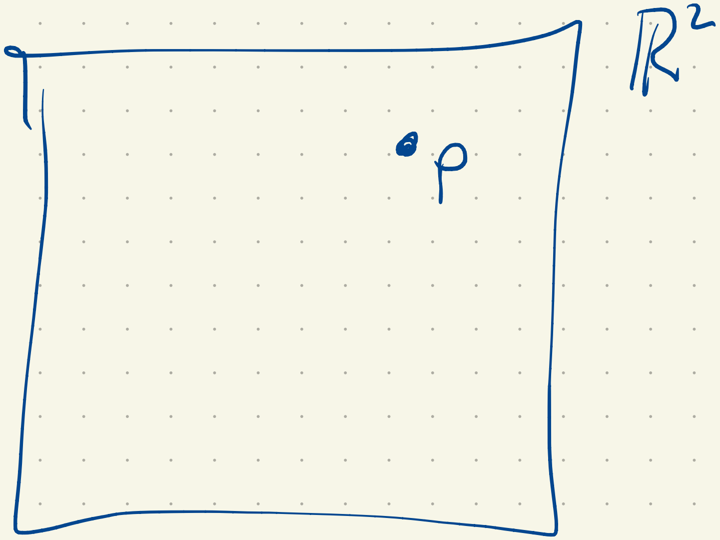
$\mathbb{R} \times I$

$g(x) = 0$

$H(x,0) = x \cdot (1-0) = x$

$H(x,1) = x \cdot (1-1) = 0$





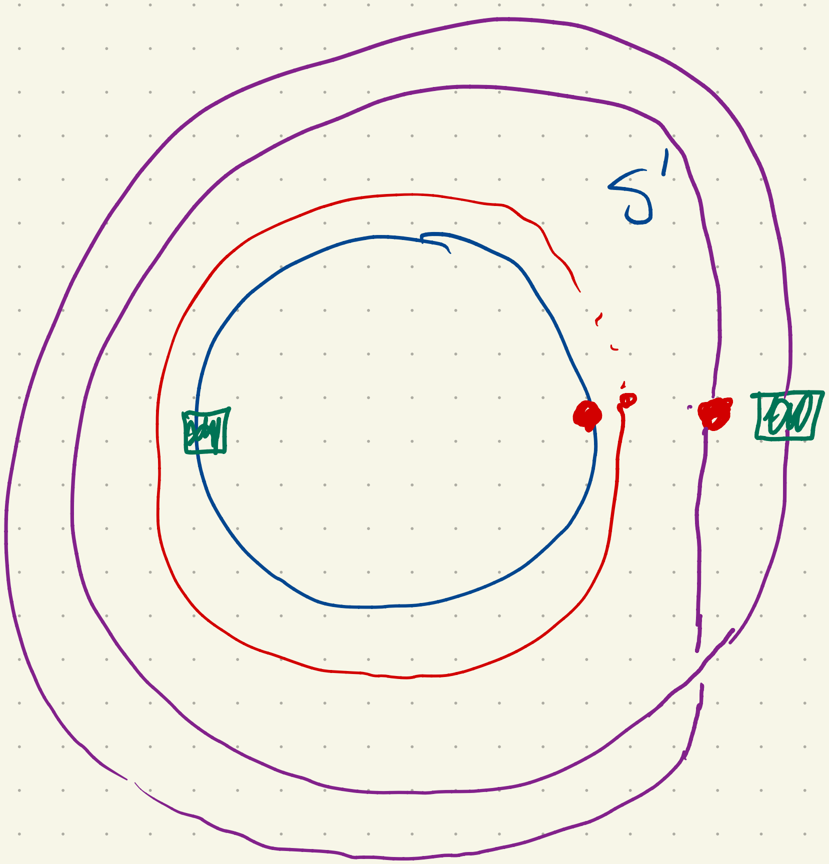
id

c_p

$c_p(x) = p \quad \forall x$

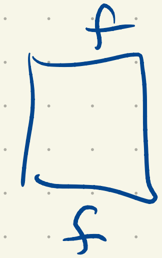
$$H(x,t) = x(1-t) + pt$$

"The identity map on $\mathbb{R}^2$ is homotopic to a constant"



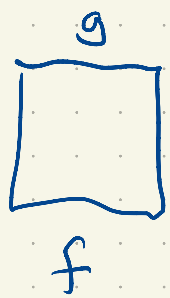
We'll see that $id : S^1 \to S^1$ is not homotopic to a constant.

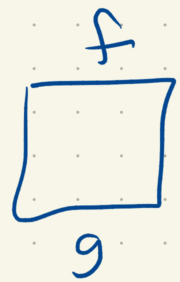
Homotopy defines an equivalence relation on the continuous functions $X \to Y$.



$f \sim f \qquad H(x,t) = f(x)$

If $f \sim g$ then $g \sim f$

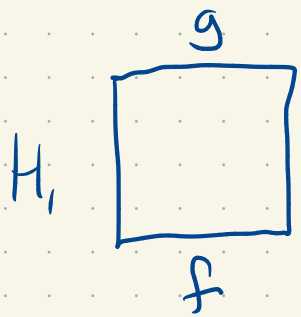
H 

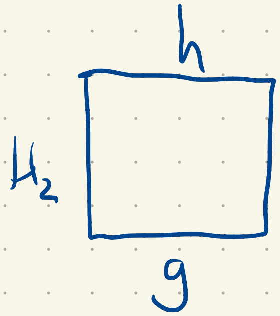
G 

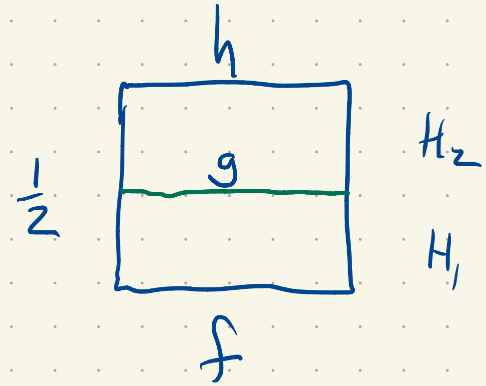
$$G(x,t) = H(x, 1-t)$$

$$(x,t) \to (x, 1-t) \to H(x, 1-t)$$

If $f \sim g$ and $g \sim h$ is $f \sim h$?







$$H(x,t) = \begin{cases} H_1(x, 2t) & 0 \le t \le \frac{1}{2} \to H_1(x,1) = g(x) \\ H_2(x, 2t-1) & \frac{1}{2} \le t \le 1 \to H_2(x,0) = g(x) \end{cases}$$

By Gluing Lemma, H is continuous.