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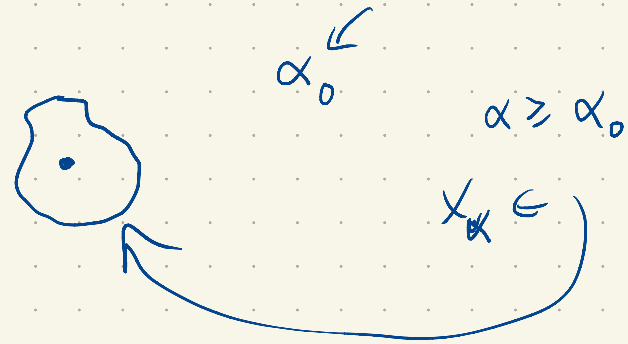
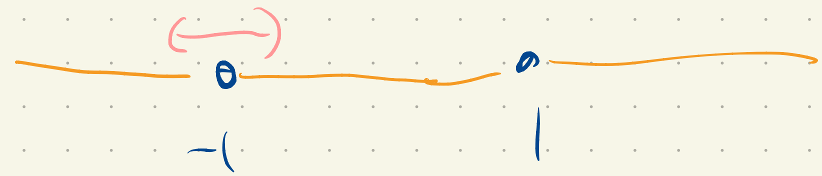


$x_n = (-1)^n$



We say that a net $\langle x_\alpha \rangle_{\alpha \in A}$ is <u>frequently</u> in a set W

if for all $\alpha_0 \in A$ there exists $\alpha \geq \alpha_0$ with $x_\alpha \in W$.

Prop: Let X be a top space and let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X. Then $x \in X$ is a cluster point of the net iff there exists a subnet converging to x.

Pf: Suppose $\langle x_{\alpha_\beta} \rangle_{\beta \in B}$ is a subnet converging to some x, of the original net $\langle x_\alpha \rangle_{\alpha \in A}$.

We wish to show x is a cluster point. Consider an open set U containing x and some index $\alpha_0 \in A$.

We need to show that there exists $\alpha \geq \alpha_0$ with $x_\alpha \in U$.

Pick β_1 in B with $\alpha_{\beta_1} \geq \alpha_0$ (cofinality).

Pick β_2 in B such that if $\beta \geq \beta_2$ then $x_{\alpha_\beta} \in U$ (convergence).

Pick β_3 in B with $\beta_3 \geq \beta_1$ and β_2 (directedness).

I claim $x_{\alpha_{\beta_3}} \in U$ and $\alpha_{\beta_3} \geq \alpha_0$.

Indeed $x_{\alpha_{\beta_3}} \in U$ since $\beta_3 \geq \beta_2$.

Moreover $\beta_3 \geq \beta_1$ so $\alpha_{\beta_3} \geq \alpha_{\beta_1} \geq \alpha_0$. (increasing).

Conversely, suppose x is a cluster point of $\langle x_\alpha \rangle_{\alpha \in A}$.

Job: find a subnet converging to x.

Consider $B = \{ (U, \alpha) \in \mathcal{U}(x) \times A : \ x_\alpha \in U \}$.

We make this a directed set via

$$(U_1, \alpha_1) \geq (U_2, \alpha_2) \text{ if } U_1 \subseteq U_2 \text{ and } \alpha_1 \geq \alpha_2.$$

Given (U_1, α_1) and (U_2, α_2) in B let $U_3 = U_1 \cap U_2$.

Pick $\hat{\alpha}$ with $\hat{\alpha} \geq \alpha_1, \alpha_2$. Now pick $\alpha_3 \geq \hat{\alpha}$ $\quad (U_3, \hat{\alpha}) \notin B$

with $x_{\alpha_3} \in U_3$. Then $(U_3, \alpha_3) \in B$ and $\quad x_{\hat{\alpha}} \notin U_3$

$(U_3, \alpha_3) \geq (U_i, \alpha_i) \quad i = 1, 2.$

Consider the map $(U, \alpha) \longrightarrow \alpha$.

This is clearly increasing. It is cofinal because it's surjective ($(X, \alpha) \in B$ for all $\alpha \in A$). Hence we have a subnet $\langle x_{\alpha_\beta} \rangle_{\beta \in B}$.

We claim $x_{\alpha_\beta} \to x$. Let W be open about x.

Pick γ with $x_\gamma \in W$; such a γ exists since x is a cluster point. Suppose $(U, \alpha) \geq \underbrace{(W, \gamma)}_{\in B}$.

Then $x_{\alpha_{(U,\alpha)}} \in U \subseteq W$.

$\square$

Prop: A top space X is compact iff every net in X has a cluster point.

Pf. Let X be compact and let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X. Let $F_\alpha = \overline{\{x_\beta : \beta \geq \alpha\}}$. The sets F_α are closed and satisfy the finite intersection property. Indeed, give $F_{\alpha_1}, \dots, F_{\alpha_n}$ we can find $\alpha^* \geq \alpha_1, \dots, \alpha_n$ and $x_{\alpha^*} \in F_{\alpha_i}$ $i = 1, \dots, n$.

Since X is compact, $\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$. Pick some x in the intersection. I claim x is a cluster point. Let O be open about x and let $\alpha_0 \in A$. Since $x \in F_{\alpha_0} = \overline{\{x_\alpha : \alpha \geq \alpha_0\}}$ it is a contact point of $\{x_\alpha : \alpha \geq \alpha_0\}$. Since U is open about x it contains an element of . I.e. U contains x_α for some $\alpha \geq \alpha_0$.

Conversely suppose X is not compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X with no finite subcover. Let $\mathcal{B}$ be the set of all finite subsets of A. For each $\beta \in \mathcal{B}$ (so $\beta = \{\alpha_1, \dots, \alpha_n\}$) pick x_β with $x_\beta \notin \bigcup_{i=1}^{n} U_{\alpha_i}$. Note: $\mathcal{B}$ is a directed set ordered by inclusion: $\beta_1 \geq \beta_2$ if $\beta_1 \supseteq \beta_2$.

Consider the net $\langle x_\beta \rangle_{\beta \in \mathcal{B}}$. Let $x \in X$.

To see that x is not a cluster point pick α_0 such that $x \in U_{\alpha_0}$. Suppose $\beta \geq \{\alpha_0\} \in \mathcal{B}$. Than $\alpha_0 \in \beta$ and hence $x_\beta \notin U_{\alpha_0}$.

So x is not a cluster point.

Summary: X is compact $\Leftrightarrow$ every net in X has a convergent subnet.

Compact Hausdorff spaces are fantastic.

Next best thing: locally compact Hausdorff spaces.

Def: A space X is locally compact if for all $x \in X$ there exists an open set U and a compact set K such that $x \in U \subseteq K$.

$\overline{U}$

We kinda want U to be closed.

We say a set $A \subseteq X$ is <u>pre compact</u> if $\overline{A}$ is compact.

There's no solid relationship between closure and compactness however unless we assume something additional about X,

We'll assume that it is Hausdorff,

In a locally compact Hausdorff space, each x in X has an open set U about it with $\overline{U}$ compact.

(every point has a precompact neighborhood).

$x \in U \subseteq K$, K is cpct $\Rightarrow$ closed

$\overline{U} \subseteq K$

$\overline{U}$ is a closed subset of a compact space and hence cpct.

Prop: Let X be a Hausdorff space. Then TFAE:

$x \in U \subseteq \overline{U} = K$

1) X is locally compact.

2) For all $x \in X$ there is a precompact open set containing x

3) X admits a basis of precompact sets.

Pf: 3) $\Rightarrow$ 2) $\Rightarrow$ 1) are all easy.

We'll show 1) $\Rightarrow$ 3).

Let $\mathcal{B} = \{ B \subseteq X : B$ is <u>open</u> and $\overline{B}$ is compact$\}$

To see that $\mathcal{B}$ is a basis let $x \in X$ and let U be an open set containing x. Since $\mathcal{B}$ consists ~~of~~ open sets, to show $\mathcal{B}$ is a basis it suffices to show there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$.

Pick some $\hat{B} \in \mathcal{B}$ with $x \in \hat{B}$; this is possible since X is LCH. Let $B = U \cap \hat{B}$. Clearly $x \in B$.

Moreover $\overline{B} = \overline{U \cap \hat{B}} \subseteq \overline{\hat{B}}$ which is compact as $\hat{B} \in \mathcal{B}$.

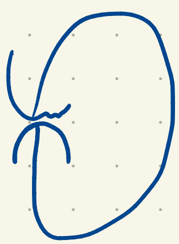
So $\overline{B}$ is a closed subset of a compact space and is cpt.

Hence $B \in \mathcal{B}$ and $x \in B \subseteq U$.

$\mathbb{R}$

$x \in \mathbb{R}$

$\rightarrow (x-1, x+1)$



$\mathbb{R}$ "$\subseteq$" S^1

$\rightarrow$ compact hausdorff space.