Friday 10pm

A net is a function from a $\underbrace{\text{directed}}$ set $A \to X$

$\leq$ $\quad \alpha \leq \alpha \quad \forall \alpha \in A$

transitivity

$\forall \alpha, \beta \in A$ there exists $\gamma \in A$ $\gamma \geq \alpha$, $\gamma \geq \beta$

X, top space $\quad x \in X \quad \mathcal{U}(x)$

$U, V \in \mathcal{U}(x) \quad U \geq V$ if $U \subseteq V$

Convergence of nets: $\quad x_\alpha \to x$ if for all $U \in \mathcal{U}(x)$,

there exists $\alpha_0 \in A$ s.t. if $\alpha \geq \alpha_0$, $x_\alpha \in U$.

Next HW: Nets have unique limits iff the space is Hausdorff.

$V \subseteq X$, $x \in \overline{V}$ iff there exists a net in V converges to x

X, Y $f: X \to Y$, f is continuous iff whenever
$x_\alpha \to x$ in X, $f(x_\alpha) \to f(x)$ in Y.

Today: characterize compactness using nets.

Recall: a metric space is compact iff it is $\underbrace{\text{sequentially compact}}$

$\longrightarrow$ $\underbrace{\text{every sequence has a convergent subsequence}}$

every net has a convergent $\boxed{\text{subnet}}$ $\longrightarrow$?

Def: Let $\{F_\alpha\}_{\alpha \in I}$ be a colletion of subsets of some set X. We say the collection has the <u>finite intersection property</u> if for any finite collection of indices $\alpha_1, \ldots, \alpha_n$, $\bigcap_{j=1}^{n} F_{\alpha_j} \neq \phi$.

Prop: A topological space X is compact iff whenever $\{F_\alpha\}_{\alpha \in I}$ is a collection of closed sets in X with the FIP, $\bigcap_{\alpha \in I} F_\alpha \neq \phi$.

$X = (0, \infty)$ $\quad F_n = (0, \frac{1}{n}]$ closed in X.

$\{F_n\}$ satisfies the FIP but $\bigcap F_n = \phi$.

$\{F_\alpha\}_{\alpha \in I}$ $\qquad$ $U_\alpha = F_\alpha^c$

$\hookrightarrow$ open

$\{U_\alpha\}_{\alpha \in I}$ is an open cover $\iff \displaystyle\bigcup_{\alpha \in I} U_\alpha = X$

$$\iff \left(\bigcup_{\alpha \in I} U_\alpha\right)^c = X^c$$

$$\iff \bigcap_{\alpha \in I} U_\alpha^c = \emptyset$$

$$\iff \bigcap_{\alpha \in I} F_\alpha = \emptyset$$

There exists a finite subcover $\iff \displaystyle\bigcup_{i=1}^{n} U_{\alpha_i} = X$

$$\iff \bigcap_{i=1}^{n} F_{\alpha_j} = \emptyset$$

Subnets

Let A, B be directed sets.

We say $f : B \to A$ is

- increasing if wherever $\beta_1 \leq \beta_2$ in B, $f(\beta_1) \leq f(\beta_2)$ in A.
- cofinal if for all $\alpha \in A$ there exists $\beta \in B$ with $f(\beta) \geq \alpha$.

Def: Let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net. A subnet of this net is a net of the form $\langle x_{f(\beta)} \rangle_{\beta \in B}$ where $f : B \to A$ is increasing and cofinal.

$f(\beta) \leftrightarrow \alpha_\beta$ $\quad \langle x_{\alpha_\beta} \rangle_{\beta \in B}$.

$A, B = \mathbb{N}$

$f : B \to A$ increasing, cofinal

not necessarily strictly increasing

cofinal $\Leftrightarrow$ $f(B)$ is not bounded above.

$1 \to 1,\ 2 \to 1,\ 3 \to 1,\ 4 \to 4,\ 5 \to 5,\ 6 \to 6, \ldots$

f

$\{x_n\}_{n=1}^{\infty}$

$x_1, x_1, x_1, x_4, x_5, x_6, \ldots$

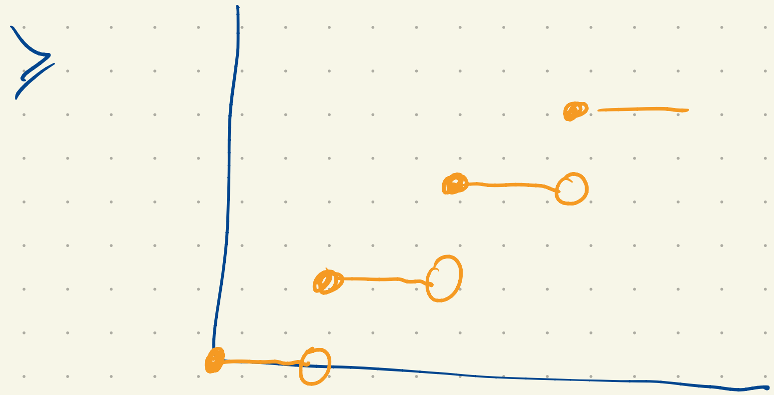
Subsequences of sequences are subnets.

(strictly increasing $\Rightarrow$ increasing cofinal)

Subnets of sequences need not be subsequences.

$$\mathbb{R}_{\geq 0} \longrightarrow \mathbb{N}$$

$$z \longrightarrow \lfloor z \rfloor$$



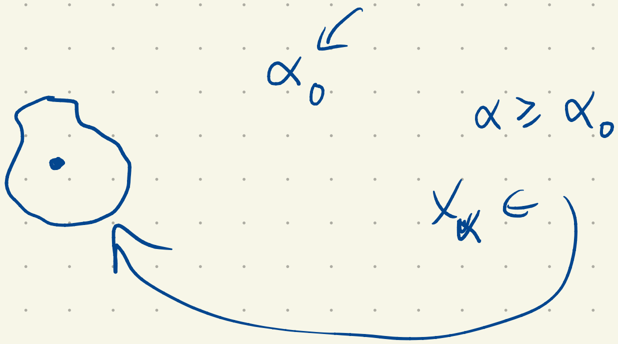
$$\langle x_n \rangle_{n \in \mathbb{N}}$$

$$\langle x_{\lfloor z \rfloor} \rangle_{z \in \mathbb{R}_{\geq 0}} \longleftarrow \text{subnet.} \quad (!)$$

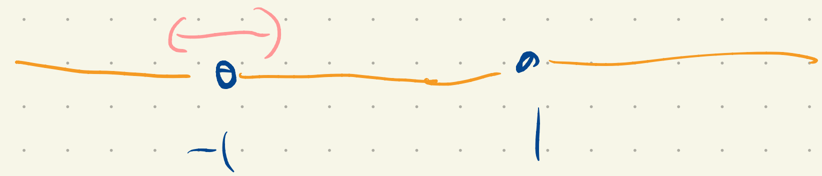
Defi: Let X be a top. space and let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X. We say $x \in X$ is a <u>cluster point</u> of the net if for every open set U containing x and every $\alpha_0 \in A$ there is $\alpha \geq \alpha_0$ with $x_\alpha \in U$.

45

1000



$x_n = (-1)^n$



We say that a net $\langle x_\alpha \rangle_{\alpha \in A}$ is <u>frequently</u> in a set W if for all $\alpha_0 \in A$ there exists $\alpha \geq \alpha_0$ with $x_\alpha \in W$.

Prop: Let X be a top space and let $\langle x_\alpha \rangle_{\alpha \in A}$ be a net in X. Then $x \in X$ is a cluster point of the net iff there exists a subnet converging to x.

Pf: Suppose $\langle x_{\alpha_\beta} \rangle_{\beta \in B}$ is a subnet converging to some x, of the original net $\langle x_\alpha \rangle_{\alpha \in A}$.

We wish to show x is a cluster point. Consider an open set U containing x and some index $\alpha_0 \in A$.

We need to show that there exists $\alpha \geq \alpha_0$ with $x_\alpha \in U$.

Pick β_1 in B with $\alpha_{\beta_1} \geq \alpha_0$ (cofinality).

Pick β_2 in B such that if $\beta \geq \beta_2$ then $x_{\alpha_\beta} \in U$ (convergence).

Pick β_3 in B with $\beta_3 \geq \beta_1$ and β_2 (directedness).

I claim $x_{\alpha_{\beta_3}} \in U$ and $\alpha_{\beta_3} \geq \alpha_0$.

Indeed $x_{\alpha_{\beta_3}} \in U$ since $\beta_3 \geq \beta_2$.

Moreover $\beta_3 \geq \beta_1$ so $\alpha_{\beta_3} \geq \alpha_{\beta_1} \geq \alpha_0$. (increasing).

Conversely, suppose x is a cluster point of $\langle x_\alpha \rangle_{\alpha \in A}$.

Job: find a subnet converging to x,

Consider $B = \{ (U, \alpha) \in \mathcal{U}(x) \times A : \ x_\alpha \in U \}$.

We make this a directed set via

$$(U_1, \alpha_1) \geq (U_2, \alpha_2) \text{ if } U_1 \subseteq U_2 \text{ and } \alpha_1 \geq \alpha_2.$$

Given (U_1, α_1) and (U_2, α_2) in B let $U_3 = U_1 \cap U_2$.

Pick $\hat{\alpha}$ with $\hat{\alpha} \geq \alpha_1, \alpha_2$. Now pick $\alpha_3 \geq \hat{\alpha}$ $(U_3, \hat{\alpha}) \notin B$ with $x_{\alpha_3} \in U_3$. Then $(U_3, \alpha_3) \in B$ and $x_{\hat{\alpha}} \notin U_3$

$(U_3, \alpha_3) \geq (U_i, \alpha_i)$ $i = 1, 2$.

$(U, \alpha) \longrightarrow \alpha \longrightarrow$ increasing, cofinal

$$\boxed{\langle x_\alpha \rangle_{(U,\alpha) \in B} \longrightarrow x}$$