Quotient spaces are awful.

A quotient of Hausdorff spaces need not be Hausdorff.

- - - - Loc Euc. — - - — - Loc Euc. - -

— - - manifold ——— - - a manifold

If $\mathcal{B}$ is a basis for X, $\{\pi(B) : B \in \mathcal{B}\}$ need not be a basis for $X/\sim$.

Lemma: A quotient of a Lindelöf space is Lindelöf.

$\mathbb{RP}^n$ $\quad \mathbb{R}^{n+1,*}/\sim$

Exercise: If $\pi : X \to Y$ is a quotient map and X is 2^{nd} countable and Y is locally Euclidean then Y is 2^{nd} countable.

(loc Euc + Lind $\Rightarrow$ 2^{nd} countable!)

Pf of Lemma:

Let $\{U_\alpha\}$ be an open cover of Y.

Since X is Lindelöff we can reduce the open cover $\{\pi^{-1}(U_\alpha)\}$ of X to a countable subcover $\{\pi^{-1}(U_{\alpha_k})\}_{k\in\mathbb{N}}$

Then $Y = \pi(X) = \pi\left(\bigcup_k \pi^{-1}(U_{\alpha_k})\right) = \pi\left(\pi^{-1}\left(\bigcup_k U_{\alpha_k}\right)\right)$ (surjective!)

$= \bigcup_k U_{\alpha_k}.$

Connectedness

Def: Let X be a top space. A <u>separation of</u> X is a pair of disjoint, non empty open sets U, V such that $U \cup V = X$. A space is disconnected if it admits a separation. Otherwise it is <u>connected</u>.

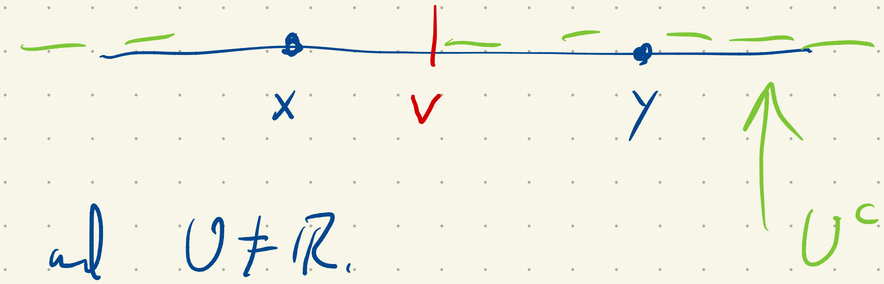
E.g: $\mathbb{Z}$ is disconnected.

$\{0\}, \ \mathbb{Z} \setminus \{0\}$ is a separation.

$\mathbb{Q}$ is disconnected

$U = \mathbb{Q} \cap (-\infty, \sqrt{2}) \quad V = \mathbb{Q} \cap (\sqrt{2}, \infty)$ is a separation.

Prop: $\mathbb{R}$ is connected.



Pf: Suppose $U \subseteq \mathbb{R}$ is open, $U \neq \emptyset$ and $U \neq \mathbb{R}$.

We need to show that U^c is not open.

Pick $x \in U$ and $y \in U^c$. We will assume $x < y$;

the other case is proved similarly.

Let $W = \{ w \in U^c : x < w \}$. Then W is nonempty (it contains y)

and bounded below by x and hence admits an infimum v.

$(v-\varepsilon, v+\varepsilon)$

From elementary analysis every interval $[v, v+\varepsilon)$ for $\varepsilon > 0$

intersects W (for otherwise $v + \varepsilon/2$ is a lower bound for W).

Hence $v \notin U$ as U is open. Hence $x < v$ since $x \leq v$

(x is a lower bound for W) and since $x \neq v$.

But then $[x, v) \subseteq U$ for otherwise v is not a lower bound for W. But then every interval $(v-\varepsilon, v+\varepsilon)$ intersects U

so v is not in the interior of U^c and U^c is not open.

Connectedness is clearly topological (it is preserved by homeos)

Cor: Open intervals in $\mathbb{R}$ are connected.

Prop: If Y is connected and $f: X \to Y$ is continuous and

surjective then Y is connected.

Pf. Suppose $f: X \to Y$ is continuous and Y is disconnected.

Let U, V be a separation of Y.

Then $f^{-1}(U)$, $f^{-1}(V)$ are

- open (continuity)
- nonempty (surjectivity)

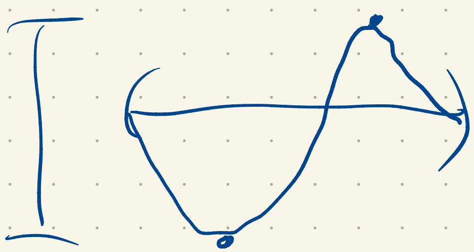
• disjoint (set theoretic)

Moreover, $f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(Y) = X$.

So X is disconnected.

Cor: The image of a connected space under a continuous map is connected.

Cor: $[-1,1]$ is connected. s.h.



$A \subseteq B \subseteq \overline{A}$

connected connected

Note: A space X is connected iff the only subsets of X that are both open and closed are X and ϕ.

Prop: If $A \subseteq X$ is connected and if U and V are disjoint open sets in X such that $A \subseteq U \cup V$ then $A \subseteq U$ or $A \subseteq V$.

Pf: If $A \cap V$ and $A \cap U$ are both nonempty then they form a separation of A. So either $A \subseteq V$ or $A \subseteq U$.

Exercise: is ϕ connected?

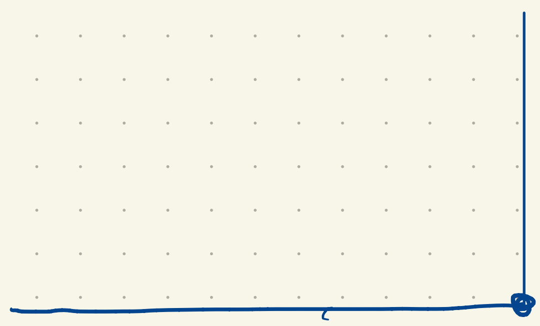
Prop: Suppose $\{A_\alpha\}_{\alpha\in I}$ is a collection of connected sets in X. If $\bigcap A_\alpha \neq \phi$ then $\bigcup A_\alpha$ is connected.

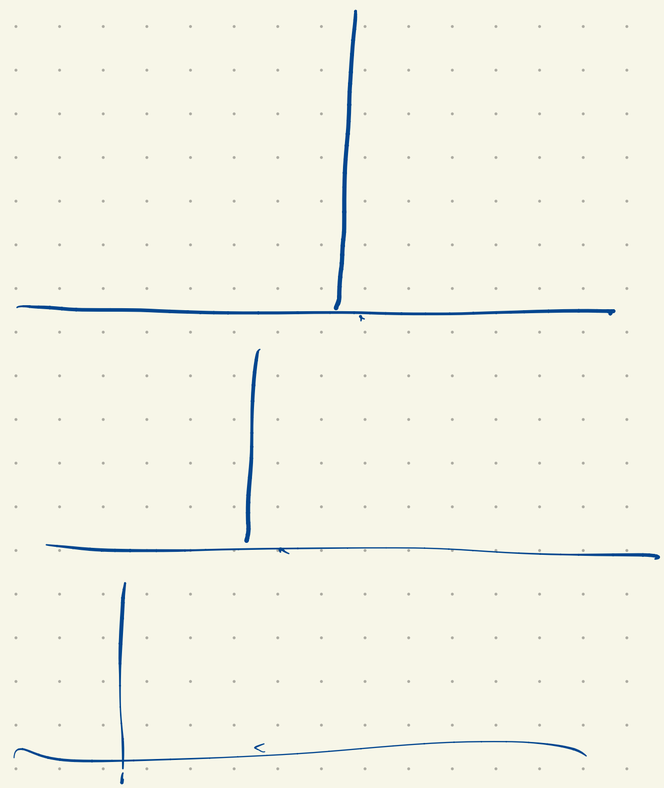
[the union of connected sets with a single point in common is connected]

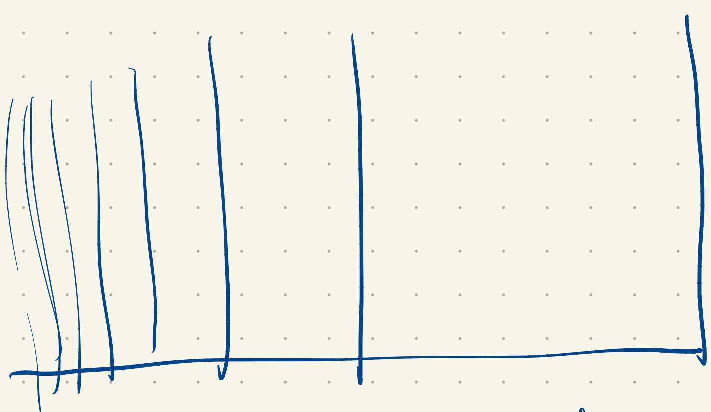
Pf: Let $A = \bigcup A_\alpha$. Suppose U, V are disjoint open sets in A with $A = U \cup V$. We need to show that one of U or V is A and the the other is therefore empty.

Each A_α is connected and is ~~therefore~~ either contained in U or in V. But all the A_α's have a point in common and hence must all be contained in the same option of

U or V, say U. Then $A \subseteq U \subseteq A$, so $U = A$ and $V = \emptyset$.







comb space