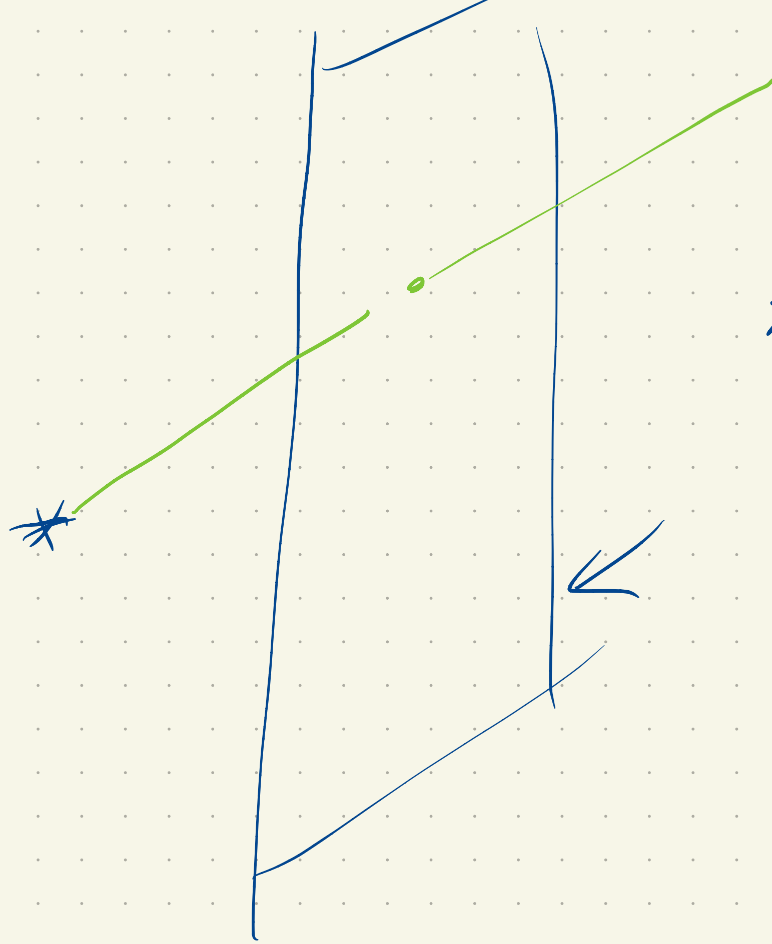
$$\begin{tikzcd}
\mathbb{R}^{n+1,*} \arrow[r, "p"] \arrow[d, "\pi_1"'] & S^n \arrow[r, "i"] \arrow[d, "\pi_2"] & \mathbb{R}^{n+1,*} \arrow[d, "\pi_1"] \\
\mathbb{R}P^n \arrow[r, "g"'] & S^n/\sim \arrow[r, "f"'] & \mathbb{R}P^n
\end{tikzcd}$$

$\pi_1(\lambda x) = \pi(x)$
if $\lambda \neq 0$

$$\begin{aligned}
f(g(\pi_1(x))) &= \pi_1(i(p(x))) \\
&= \pi_1\left(i\left(\frac{x}{\|x\|}\right)\right) \\
&= \pi_1\left(\frac{x}{\|x\|}\right) \\
&= \pi_1(x)
\end{aligned}$$

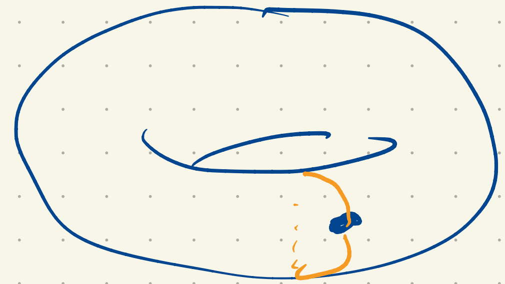
$$\begin{tikzcd}
S^n \arrow[r, hook, "i"] \arrow[d, "\pi_2"'] & \mathbb{R}^{n+1,*} \arrow[r, "p"] \arrow[d, "\pi_1"] & S^n \arrow[d, "\pi_2"] \\
S^n/_{\sim} \arrow[r, "f"'] & \mathbb{R}P^n \arrow[r, "g"'] & S^n/_{\sim}
\end{tikzcd}$$

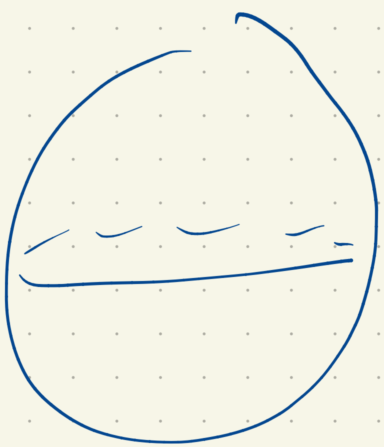
$$\begin{aligned}
g(f(\pi_2(x))) &= \pi_2(p(i(x))) \\
&= \pi_2(p(x)) \\
&= \pi_2\left(\frac{x}{\|x\|}\right) \\
&= \pi_2(x) \qquad (\|x\| = 1 \,!)
\end{aligned}$$

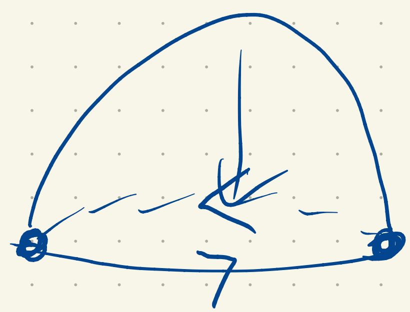


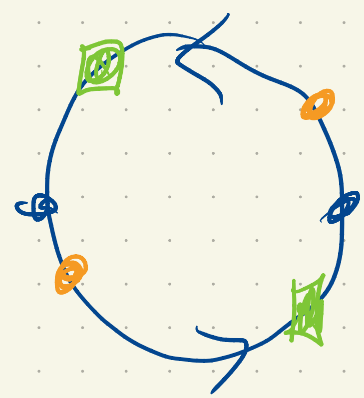
$(x_1, x_2, \dots, x_n, 1)$

$x_{n+1} = 1$









Quotient Map.

$$\underset{\text{space}}{X} \xrightarrow[\text{surjection}]{\pi} \underset{\text{set}}{Y}$$

Def: The quotient topology on Y <u>induced</u> by π is

$$\{ U \subseteq Y : \pi^{-1}(U) \text{ is open in } X \}.$$

Exercise: this is a top..

If X has an equiv. rel.

$$X \xrightarrow{\pi} X/\sim$$

The quotient topology induced by π is the similar quotient top. already defined.

Note: If $\pi : X \to Y$ is a surjection then

X acquires an equiv. rel $x_1 \sim x_2$ iff $\pi(x_1) = \pi(x_2)$.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "\pi"] & \\ X/\sim & & Y \end{tikzcd}$$

If Y has the quotient top. we'll shortly see $Y \sim X/\sim$.

If X and Y are spaces and $\pi : X \to Y$ is a surjection we say π is a <u>quotient</u> <u>map</u> if the topology on Y is the quotient topology induced by π.

map $\longleftrightarrow$ cts.

Prop A map $\pi: X \to Y$ is a quotient map if and only if it is surjective, continuous, and takes saturated open sets to open sets.

Cor: Surjective cts, open function are quotient maps.

Pf: Suppose π is a quotient map. It is continuous and surjective. Suppose $U \subseteq X$ is saturated and open. Then $U = \pi^{-1}(W)$ for some $W \subseteq Y$. Moreover, W is open in Y since Y has the quotient topology. But $\pi(U) = \pi(\pi^{-1}(W)) = W$ since π is a surjection. So π takes saturated open sets to open sets.

Conversely, suppose π is continuous, surjective and takes sat. open sets to open sets. To show π is a q.m. we need to show for all $A \subseteq Y$, $\pi^{-1}(A)$ is open iff A is open. Well, if $A \subseteq Y$ is open then $\pi^{-1}(A)$ is open by continuity. On the other hand, suppose $\pi^{-1}(A)$ is open. It is also saturated. So $\pi(\pi^{-1}(A))$ is open as π takes sat. open sets to open sets. But $\pi(\pi^{-1}(A)) = A$, so A is open.

Exercise: $A \subseteq X/\sim$ is closed iff $\pi^{-1}(A)$ is closed and similarly for quotient maps.

HW: A continuous surjection π is a quotient map iff it takes saturated closed sets to closed sets.

Cor: A surjective, cts, closed function is a quotient map.

$$[0,1] \longrightarrow S^1$$
$$x \overset{\mathcal{E}}{\longmapsto} e^{2\pi i x}$$

Exercise in analysis: $\mathcal{E}$ is closed.

$\mathcal{E}$ is also a surjection,

So $\mathcal{E}$ is a quotient map.

$[0,1]$, $\sim$ $\quad 0 \sim 1$

$$\begin{tikzcd}
 & {[0,1]} \arrow[dl, "\pi"'] \arrow[dr, "\mathcal{E}"] & \\
{[0,1]/\sim} & & S^1
\end{tikzcd}$$

Thm Uniqueness of Quotients

Suppose $\pi_i : X \to Y_i$ $i = 1, 2$ are quotient maps that make the same identifications. I.e., $\pi_1(x_1) = \pi_1(x_2)$ if and only if $\pi_2(x_1) = \pi_2(x_2)$. Then Y_1 is homeomorphic to Y_2 by the map taking any $\pi_1(x)$ to $\pi_2(x)$.

Pf:

$$\begin{tikzcd} X \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y_1 \arrow[r, dashed, "f_{1,2}"'] & Y_2 \end{tikzcd}$$

Since π_2 makes the same identifications as π_1, it descends to the quotient as a continuous map $f_{1,2}$.

A similar argument shows π_1 descends to a continuous $f_{2,1} : Y_2 \to Y_1$.

Moreover $f_{2,1}(f_{1,2}(\pi_1(x))) = f_{2,1}(\pi_2(x))$

$= \pi_1(x)$.

$$\begin{tikzcd} X \arrow[d, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ Y_2 \arrow[r, dashed, "f_{2,1}"'] & Y_1 \end{tikzcd}$$

Similarly $f_{1,2}(f_{2,1}(\pi_2(x))) = \pi_2(x)$

for all $x \in X$.

So $f_{2,1} = f_{1,2}^{-1}$.

Exercise: Show that the CPQT holds for quotient maps.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

π is a q.m.

If $\tilde{f}$ is cts and const on fibers of π, it descends to a cts map