Last class: QT

X, $\sim$, $\pi : X \to X/\sim$

space

$A \subseteq X/\sim$ is open iff $\pi^{-1}(A)$ is open in X

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ X/\sim \arrow[r, "f"'] & Z \end{tikzcd}$$

f is continuous iff $\tilde{f}$ is cts.

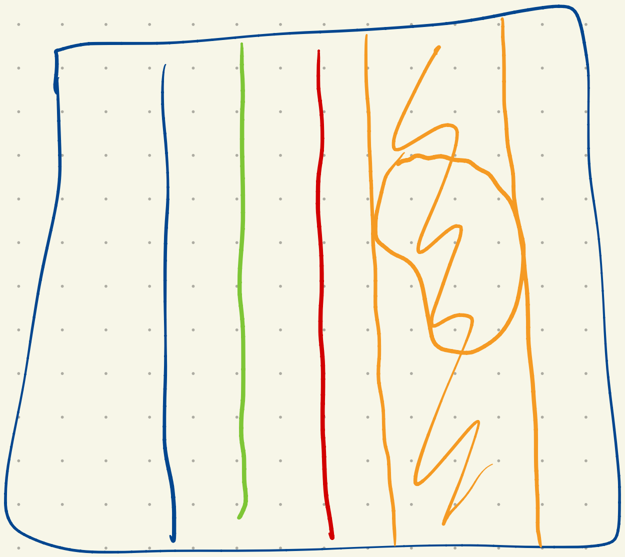
Recall: We "represent" points in $X/\sim$ by fibers $\pi^{-1}(\{x\})$

We "represent" sets in $X/\sim$ by saturated sets $\pi^{-1}(A)$.

If you want to build a function on $X/\sim$ instead you build a function on X that is constant on the fibers of π.

"$\tilde{f}$ descends to the quotient"

E.g. $\mathbb{R}^2/\sim$ $\quad (x,y_1)\sim(x,y_2) \quad$ $\mathbb{R}^2/\sim \;\sim\; \mathbb{R}$



$\mathbb{R}^2$

$$\pi(x,y) = [(x,y)]$$

$$\pi^{-1}(\mathbb{R}^2/\sim)$$

$$\pi_{\mathbb{R}}(x,y) = x$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[d, "\pi"'] \arrow[dr, "\pi_{\mathbb{R}}"] & \\ \mathbb{R}^2/\sim \arrow[r, "g"'] & \mathbb{R} \end{tikzcd}$$

$$\pi_{\mathbb{R}}(x,y_1) = \pi_{\mathbb{R}}(x,y_2) = x$$

↳ $\pi_{\mathbb{R}}$ is constant on the fibers

$$\begin{tikzcd} \mathbb{R} \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{R}^2 \arrow[d, "\pi"] \\ & \mathbb{R}^2/\sim \end{tikzcd}$$

$$i(x) = (x,0)$$

Is f continuous? Yes, by composition.

Job. f is g^{-1}

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{R}^2 \arrow[d, "\pi"] \\
& \mathbb{R}^2/\sim
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[d, "\pi"'] \arrow[dr, "\pi_{\mathbb{R}}"] & & \\
\mathbb{R}^2/\sim \arrow[r, "g"'] & \mathbb{R} \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{R}^2 \arrow[d, "\pi"] \\
& & \mathbb{R}^2/\sim
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{R}^2 \arrow[d, "\pi"'] \arrow[dr, "\pi_{\mathbb{R}}"] & \\
& \mathbb{R}^2/\sim \arrow[r, "g"'] & \mathbb{R}
\end{tikzcd}
$$

$$g(f(x)) = \pi_{\mathbb{R}}(i(x)) = \pi_{\mathbb{R}}(x, 0) = x$$

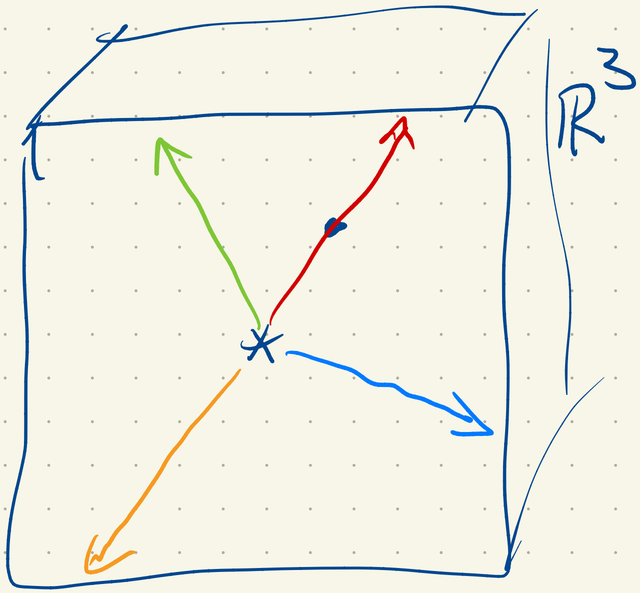
$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[d, "\pi"'] \arrow[dr, "\pi_{\mathbb{R}}"] & & \\
\mathbb{R}^2/\sim \arrow[r, "g"'] & \mathbb{R} \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{R}^2 \arrow[d, "\pi"] \\
& & \mathbb{R}^2/\sim
\end{tikzcd}
$$

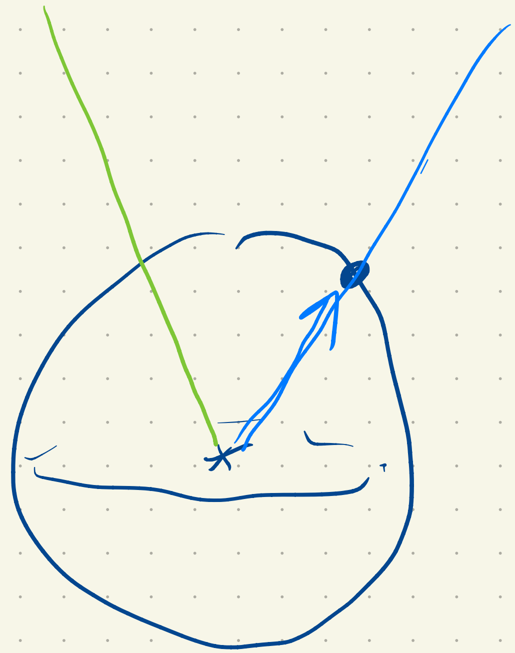
$$
\begin{aligned}
f(g(\pi(x,y))) &= \pi(i(\pi_{\mathbb{R}}(x,y))) \\
&= \pi(i(x)) \\
&= \pi(x,0) \\
&= \pi(x,y)
\end{aligned}
$$

$\Rightarrow$ if $z \in \mathbb{R}^2/\sim$ then $f(g(z)) = z$. by surjectivity of π

$\mathbb{R}^3 \setminus \{0\}$ $\quad \mathbb{R}^{3,*}$

$x \sim \lambda x \quad \lambda \in \mathbb{R}, \; \lambda > 0$





$$\begin{tikzcd}
S^2 \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{R}^{3,*} \arrow[d, "\pi"] \\
& \mathbb{R}^{3,*}/\sim
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{R}^{3,*} \arrow[d, "\pi"] \arrow[dr, "\tilde{g}"] & \\
\mathbb{R}^{3,*}/\sim \arrow[r, "g"'] & S^2
\end{tikzcd}$$

$\tilde{g}(x) = \dfrac{x}{\|x\|}$

$i(x) = x$

Is i continuous? Yep. Subspace top.

$\tilde{g}$ is evidently continuous. (no zero div.)

Is $\tilde{g}$ const on fibers of π?

x

$$\tilde{g}(\lambda x) = \frac{\lambda x}{\|\lambda x\|} = \frac{\lambda x}{|\lambda|\,\|x\|} = \frac{x}{\|x\|} = \tilde{g}(x)$$

$\lambda > 0$

So $\tilde{g}$ is const on the fibers of π and

descends to a continuous map $g: \mathbb{R}^3/\sim \to S^2$.

Claim: f and g are inverses.

$$\begin{tikzcd} S^2 \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{R}^{3,*} \arrow[d, "\pi"] \\ & \mathbb{R}^{3,*}/\sim \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^{3,*} \arrow[d, "\pi"] \arrow[dr, "\tilde{g}"] & \\ \mathbb{R}^{3,*}/\sim \arrow[r, "g"'] & S^2 \end{tikzcd}$$

$$g(f(x)) = \tilde{g}(i(x)) = \tilde{g}(x) = \frac{x}{\|x\|} = x$$

$(x \in S^2!)$

$$
\begin{tikzcd}
\mathbb{R}^{3,*} \arrow[d, "\pi"] \arrow[dr, "\tilde{g}"] & & \\
\mathbb{R}^{3,*}/\sim \arrow[r, "g"'] & S^2 \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{R}^{3,*} \arrow[d, "\pi"] \\
& & \mathbb{R}^{3,*}/\sim
\end{tikzcd}
$$

$$f(g(z)) = z$$

$\hookrightarrow \mathbb{R}^{3,*}/\sim$

$$
\begin{aligned}
f(g(\pi(x))) = \pi(i(\tilde{g}(x))) &= \pi\left(i\left(\frac{x}{\|x\|}\right)\right) \\
&= \pi\left(\frac{x}{\|x\|}\right) \\
&= \pi(x)
\end{aligned}
$$

$\mathbb{R}^{n+1,*}$ $\quad x \sim y$ if $x = \lambda y \quad \lambda \neq 0$

fibers are "lines" thru origin

$\mathbb{R}\mathbb{P}^n = \mathbb{R}^{n+1,*}/\sim$

$\hookrightarrow$ real projective space

$*$

$S^n \quad x \sim -x$

$\mathbb{R}\mathbb{P}^n \sim S^n/\sim$

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "\pi_2"'] & \mathbb{R}^{n+1,*} \arrow[d, "\pi_1"] \\ S^n/_2 \arrow[r, "f"] & \mathbb{RP}^n \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^{n+1,*} \arrow[r, "p"] \arrow[d, "\pi_1"'] & S^n \arrow[d, "\pi_2"] \\ \mathbb{RP}^n \arrow[r, "g"'] & S^n/_2 \end{tikzcd}$$

$$\pi_1\left(\frac{x}{\|x\|}\right) = \pi_1(x)$$

$$\pi_1(i(x)) \stackrel{?}{=} \pi_1(i(-x))$$
$$\downarrow$$
$$\pi_1(x) \stackrel{?}{=} \pi_1(-x) \checkmark$$

$$p(x) = \frac{x}{\|x\|}$$

p is const on fibers of π_1

so $\pi_2 \circ p$ is also,

$$\pi_1(\lambda x) = \pi_1(x) \quad \forall \lambda \neq 0$$

Exercise: $f = g^{-1}$