Exercise: Show that the CPPT holds for the product top and is characteristic.

$\mathbb{R}^\omega$, box

$\mathbb{R}^\omega$, prod.

$\mathbb{N} \to \mathbb{R}^\omega$ (sequences)

When is a sequence continuous into $\mathbb{R}^\omega$, prod?

$\{x_k\}$ $\qquad x_k = (x_k(1), x_k(2), x_k(3), \ldots)$

$x_k \longrightarrow x$

$k \longmapsto x_k \longrightarrow x_k(j)$

We have convergence $\Leftrightarrow$ we have convergence factor wise.

$$x_1 = x_1(1),\ x_1(2),\ x_1(3),\ \dots$$

$$x_2 = x_2(1),\ x_2(2),\ x_2(3),\ \dots$$

$\mathbb{R}^\omega \supseteq \ell_1,\ \ell_2,\ \ell_\infty \to \sup |x_k| < \infty$

$\ell_\infty \to$ bounded

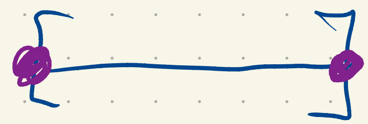
$\ell_2 \to \left(\sum |x_k|^2\right)^{1/2} < \infty$

$\ell_1 \to \sum |x_k| < \infty$

Quotient Topology.

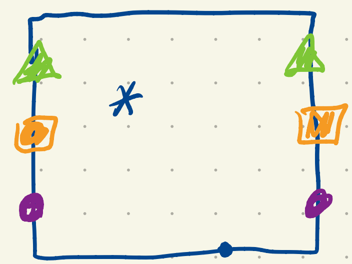
Topological spaces made by "gluing"

$I = [0,1] \qquad 0 \sim 1$



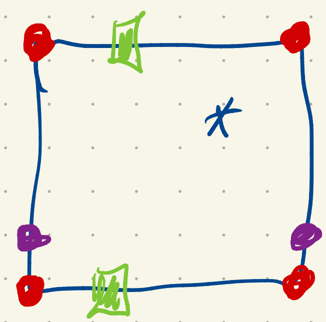
$X = I \times I \qquad (0,y) \sim (1,y)$

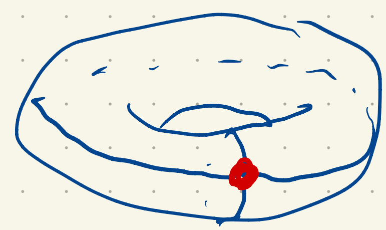
$X/\sim \;\Rightarrow$ set of equivalence classes



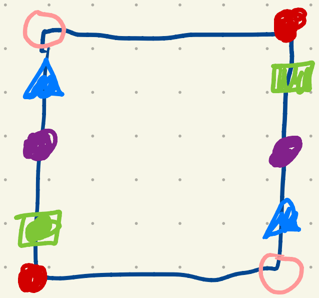
$X = I \times I \qquad (0,y) \sim (1,y)$

$(x,0) \sim (x,1)$





$X = I \times I \qquad (0,y) \sim (1, 1-y)$



Suppose $\sim$ is an equivalence relation on X.

$X/\sim$ is the set of equivalence classes

$$[x] = \{ z \in X : z \sim x \}$$

If X has a topology, is there a natural topology to put on $X/\sim$?

$$A \hookrightarrow X$$

$\pi : X \to X/\sim$ projection

$$X \times Y \xrightarrow{\pi_X} X$$

We would like π to be continuous.

We'll seek the richest possible topology on $X/\sim$ so that π is continuous.

Candidates for open sets are V where $\pi^{-1}(V)$ is open in X.

$$\begin{array}{c} X \\ \big\downarrow{\scriptstyle \pi} \\ X/\sim \end{array}$$

$$\tau = \{ V \subseteq X/\sim \,:\, \pi^{-1}(V) \text{ is open in } X \}.$$

Is τ a topology?

Is $X/\sim \in \tau$?

$$\pi^{-1}(X/\sim) = X$$
$$\pi^{-1}(\emptyset) = \emptyset$$

open

Given $\{V_\alpha\}_{\alpha \in I}$ with $\pi^{-1}(V_\alpha)$ open in X

is $\pi^{-1}\left(\bigcup V_\alpha\right)$ open in X?

$$\pi^{-1}\left(\bigcup V_\alpha\right) = \underbrace{\bigcup \left(\pi^{-1}(V_\alpha)\right)}_{\text{open in } X}$$

$$\pi^{-1}\left(\bigcap_{k=1}^{n} V_k\right) = \bigcap_{k=1}^{n} \pi^{-1}(V_k)$$

So yeah, τ is a topology, the quotient topology.

$$\begin{array}{c} X \\ \downarrow \\ X/\sim \end{array}$$

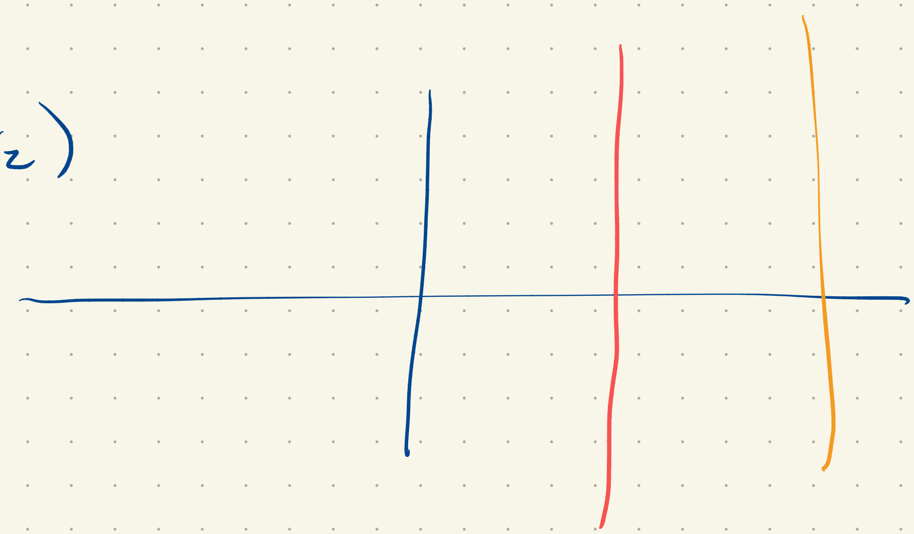
When working downstairs, think upstairs.

1) points on $X/\sim$

Def: If $x \in X/\sim$, the fiber over x is $\pi^{-1}(\{x\})$.

(it's an equivalence class!)

$\mathbb{R}^2/\sim \quad (x, y_1) \sim (x, y_2)$



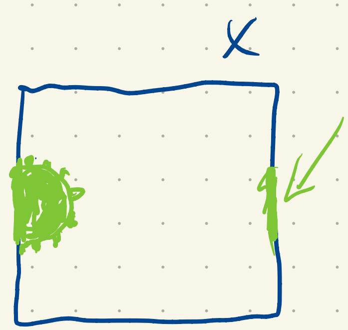
2) sets downstairs are represented by their preimages under projection

Def: A set $A \subseteq X$ is <u>saturated</u> w.r.t. π if $A = \pi^{-1}(W)$ for some set $W \subseteq X/\sim$.

Exercise: A set is saturated precisely when it's a union of fibers.

$I \times I \quad (0,y) \sim (1,y)$

V is open downstairs iff $\pi^{-1}(V)$ is open upstairs.



Recall the CPPT

$$Z \xrightarrow{f} X \times Y$$

f is cts iff $\pi_X \circ f$, $\pi_Y \circ f$ are cts.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ X/\sim \arrow[r, "f"'] & Z \end{tikzcd}$$

If f is continuous, is $\tilde{f}$ continuous?

Yes! Composition!

If $\tilde{f}$ is continuous, is f continuous?

Suppose $U \subseteq Z$ is open.

Is $f^{-1}(U)$ open?

Iff $\pi^{-1}(f^{-1}(U))$ is open in X

But $\pi^{-1}(f^{-1}(U)) = (f\circ\pi)^{-1}(U)$

$= \tilde{f}^{-1}(U)$.

Since $\tilde{f}$ is continuous, $\tilde{f}^{-1}(U)$ is open in X.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ X/\sim \arrow[r, "f"'] & Z \end{tikzcd}$$

Prop: Characteristic Property of Quotient Topology:

$f: X/\sim \to Z$ is cts iff $f\circ\pi : X \to Z$ is.

3) Functions with domain $X/\sim$ are represented by "certain" functions with domain X.

$$
\begin{tikzcd}
X \arrow[d] \arrow[dr, "\tilde{f}"] & \\
X/\sim \arrow[r, "f"'] & Z
\end{tikzcd}
$$

If $\pi(x_1) = \pi(x_2)$

we better have $\tilde{f}(x_1) = \tilde{f}(x_2)$.

We say that $\tilde{f}$ is constant on the fibers of π, if

$$\tilde{f}(x_1) = \tilde{f}(x_2) \text{ whenever}$$

$$\pi(x_1) = \pi(x_2).$$

If $\tilde{f}$ is constant on the fibers of π, there exists f that makes this diagram commute.

$$f([x]) = \tilde{f}(x)$$