$$\begin{tikzcd} X_{(r)} \arrow[r, "id_{rp}"] \arrow[dr, "\pi_j^{(r)}"'] & X_{(p)} \arrow[d, "\pi_j^{(p)}"] \\ & X_j \end{tikzcd}$$

Since $\pi_j^{(r)}$ is cts, so is id_{rp}.

for each j.

Facts: 1) A (finite) product of Hausdorff spaces is Hausdorff.

Exercise.

2) If $\mathcal{B}_1$ is a basis for X_1 and $\mathcal{B}_2$ is a basis for X_2, $\{B_1 \times B_2 : B_1 \in \mathcal{B}_1 \text{ and } B_2 \in \mathcal{B}_2\}$ is a basis for $X_1 \times X_2$.

HW

3) A product of two second countable spaces is 2^{nd} countable.

X_1, X_2, X_3 $\qquad$ $X_1 \times X_2 \times X_3$ $\qquad$ $(X_1 \times X_2) \times X_3$

(x_1, x_2, x_3) $\qquad$ $((x_1, x_2), x_3)$

HW: $(X_1 \times \cdots \times X_n) \times X_{n+1}$ is homeo to $X_1 \times \cdots \times X_{n+1}$

Use: CPPT

Want to show: A product of two manifolds is a manifold.

$M_1^{d_1}, M_2^{d_2} \sim M_1 \times M_2$

$\hookrightarrow$ dimension $d_1 + d_2$

X, Y top spaces
$\cup\!| \quad \cup\!|$
A, B subspaces

$A \times B$ has two topologies

1) subspace of product $X \times Y$

2) product of subspace topologies

HW: These are the same.

Please use the fact that CPPT is characteristic.

Hint: one of your diagrams will be a square!

Suppose X and Y are locally Euclidean with dimensions d_X and d_Y.

We wish to show that $X \times Y$ is locally Euc. w/ dim $d_X + d_Y$.

Let $(x,y) \in X \times Y$. Job: There exists an open set about (x,y) homeo to $\mathbb{R}^{d_x + d_y}$.

There exists $U_x \subseteq X$, $x \in U_x$ and $\psi : U_x \to \mathbb{R}^{d_x}$ is a homeo.

There exists $U_y \subseteq Y$ $y \in U_y$ and $\phi : U_y \to \mathbb{R}^{d_y}$ is a homeo.

Let $U = U_x \times U_y$. It contains (x,y). It's a basic open set and hence is open.

Define $\Psi : (U_x \times U_y) \to \mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$ by

$$\Psi(x,y) = (\psi(x), \phi(y)).$$

$$\psi \times \phi$$

We will shortly see that Ψ is a homeomorphism from $U_x \times U_y$ with the product top.

Exercise: For all $k, l \in \mathbb{N}$ $\mathbb{R}^k \times \mathbb{R}^l \sim \mathbb{R}^{k+l}$.

Since $U_x \times U_y$ with the product top is the same as $U_x \times U_y$ w/ subspace topology, Ψ is a homeomorphism from $U \subseteq X \times Y$ to $\mathbb{R}^{d_x + d_y}$.

Lemma: Suppose X_1, X_2, Y_1, Y_2 are top spaces and $f_i : X_i \to Y_i$ are continuous $i = 1, 2$.

Define

$$f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \text{ by}$$

$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2)),$$

Then $f_1 \times f_2$ is continuous and moreover, if each f_i is a homeomorphism then $f_1 \times f_2$ is a homeomorphism.

Pf: To show $f_1 \times f_2$ is continuous it suffice to show

$$\pi_{Y_i} \circ (f_1 \times f_2) \text{ is continuous for } i = 1, 2.$$

But $\pi_{Y_i} \circ (f_1 \times f_2) = f_i$ which is continuous.

Suppose each f_i is a homeomorphism, Then

$f_1 \times f_2$ is invertible and $(f_1 \times f_2)^{-1} = f_1^{-1} \times f_2^{-1}$

which is continuous as each f_i^{-1} is.

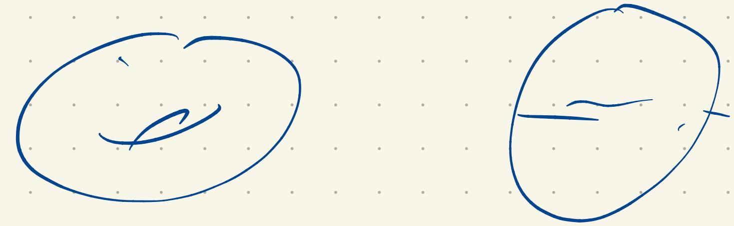
Upshot: A product of an n-manifold with an m-manifold is an $n+m$ manifold.

New manifolds S^1 S^n

$S^1 \times S^1$ is a 2-manifold (T^2, torus) dimension

$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n\text{-times}}$$

torus



Arbitrary Product.

$$\{X_\alpha\}_{\alpha \in A}$$

$$\prod_{\alpha \in A} X_\alpha = \left\{ f : A \to \bigcup_{\alpha \in A} X_\alpha : f(\alpha) \in X_\alpha \text{ for all } \alpha \in A \right\}$$

$X \times Y$ $\quad A = \{0, 1\}$

$x(k)$

x_k

$$f : \{0,1\} \to X \cup Y$$

$$\left.\begin{array}{l} f(0) \in X \\ f(1) = Y \end{array}\right] \begin{array}{l} f_0 \\ f_1 \end{array}$$

Notation $\quad X^n \quad$ Each X_α is X, $\quad A = \{0, \dots, n-1\}$

$X^\omega \quad$ Each X_α is X, $\quad A = \mathbb{N}$

(X-valued sequences)

$X^Y \quad$ X, Y are sets

X_α is X for all α

and $A = Y$.

$\{ f : Y \to X \}$

Is $\prod_{\alpha \in A} X_\alpha$ empty? No! Axiom of Choice.

Two natural choices for bases on $\prod_{\alpha \in A} X_\alpha$.

box topology

τ_b : $\prod_{\alpha \in A} U_\alpha$, $U_\alpha \subseteq X_\alpha$ is open are basic open sets

τ_p : subbasis from $\pi_\alpha^{-1}(U_\alpha)$ $U_\alpha \subseteq X_\alpha$ is open,

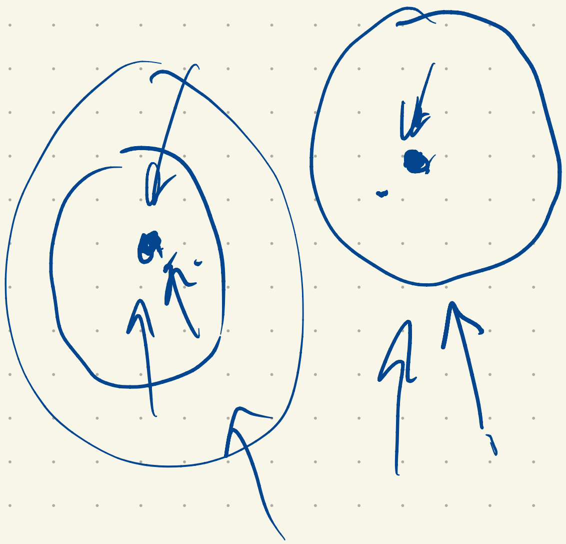
$$\bigcap_{k=1}^{n} \pi_{\alpha_k}^{-1}(U_{\alpha_k})$$ basic open sets

$U_{\alpha_1} \times \cdots U_{\alpha_n} \times X \cdots$ $\quad X$

$$\bigcap \pi_\alpha^{-1}(U_\alpha) = \prod_{\alpha \in A} U_\alpha$$

Evidently τ_b is strictly finer than τ_p if

There are infinitely many factors



T^2 T^1

$(\pi_k \circ f)(a_n)$

$\Rightarrow \pi_k \circ f(a)$

$x_n \in \mathbb{R}^k$

$x_n \to x$ $\Leftrightarrow$ $\pi_k(x_n) \to \pi_k(x)$

$f : X \to \mathbb{R}^n$

$a_n \to a$ in X $\Rightarrow$ $\underbrace{f(a_n) \to f(a) \text{ in } \mathbb{R}^n}$

$\Updownarrow$

$\pi_k(f(a_n)) \to \pi_k(f(a))$

f is cts iff $\pi_k \circ f$ is cts.