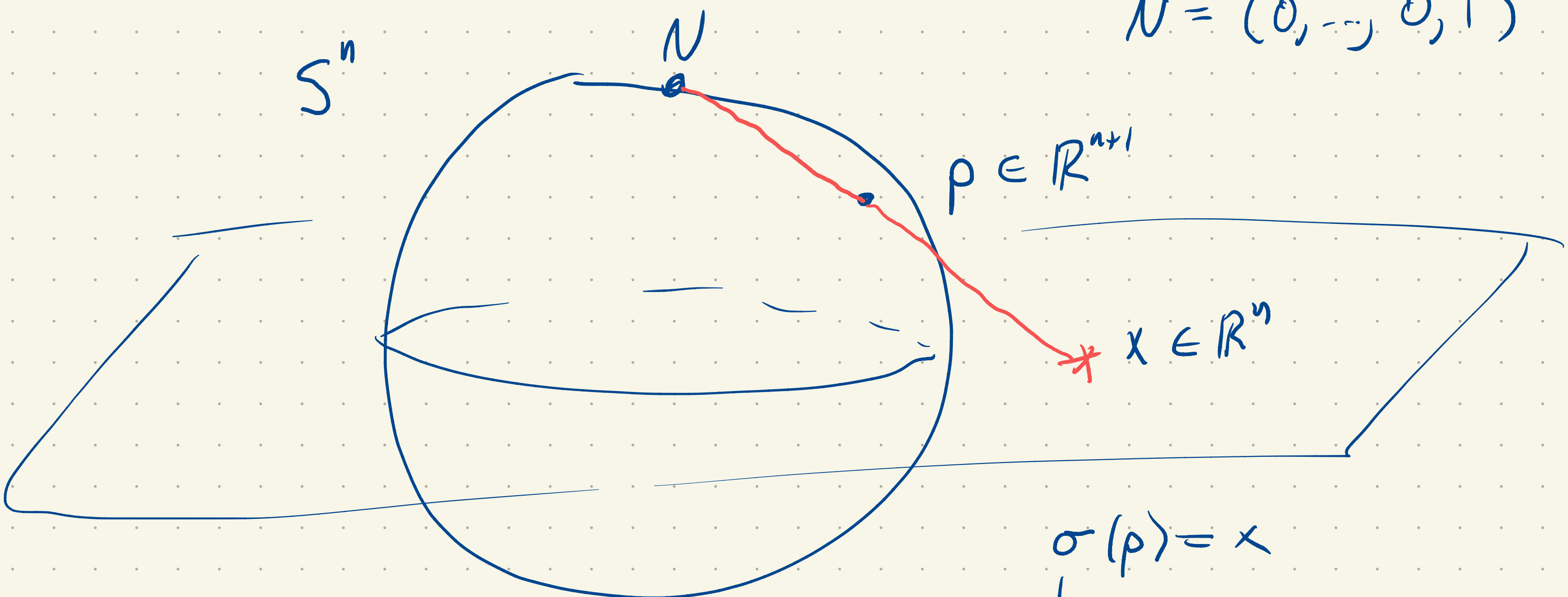
Last class: S^2 is a manifold



$N = (0, \dots, 0, 1)$

$\sigma(p) = x$ → stereographic projection.

$$\sigma(p_1, \dots, p_{n+1}) = \frac{1}{1 - p_{n+1}} (p_1, \dots, p_n)$$

$\sigma : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^n$ is continuous as it is the restriction of a continuous function.

$$\sigma^{-1}(\underbrace{x_1, \dots, x_n}_{x}) = \frac{1}{|x|^2+1}\left(2x_1, \dots, 2x_n, |x|^2-1\right)$$

σ^{-1} is continuous $\mathbb{R}^n \to \mathbb{R}^{n+1}$

image lies in S^n

so is cts into S^n.

σ is a homeomorphism

$S^n \setminus \{N\} \to \mathbb{R}^n$

Product Topology

Two top spaces X, Y.

$$X \times Y = \{(x,y) : x \in X, y \in Y\}$$

what's an ordered pair? It's a map $\{0,1\} \xrightarrow{\sigma} X \cup Y$

$\sigma(0) \in X$

$\sigma(1) \in Y$

We want a natural topology on $X \times Y$.

$$A \subseteq X \qquad \underline{A} \overset{i_A}{\hookrightarrow} X$$

$$X \times Y \xrightarrow{\pi_X} X$$

projections. We want these to be continuous.

$$X \times Y \xrightarrow{\pi_Y} Y$$

Def: A subbasis for a set X is a collection $\mathcal{S}$ of subsets of $\mathcal{S}$ such that $\bigcup \mathcal{S} = X$

$$\bigcup_{S \in \mathcal{S}} S = X$$

Given a subbasis we construct a pre-basis as follows

$$\mathcal{B} = \{ S_1 \cap \cdots \cap S_k : S_1, \ldots, S_k \in \mathcal{S} \text{ for some } k \}$$

Is this a pre basis? $\mathcal{S} \subseteq \mathcal{B}$, $\underset{=X}{\underbrace{\bigcup \mathcal{S}}} \subseteq \bigcup \mathcal{B}$

$q \in B_1 \cap B_2$ $\exists B_3$ $q \in B_3 \subseteq B_1 \cap B_2$

Our $\mathcal{B}$ is closed under $\underset{\text{finite.}}{\uparrow}$ intersections!

$\mathcal{B}$ generates a topology

- it's the smallest topology that contains $\mathcal{B}$.
- it's the intersection of all topologies that contains $\mathcal{B}$

- it consists of arbitrary unions of elements of $\mathcal{B}$,

This topology is the smallest topology that contains $\mathcal{S}$.

Why? Any topology that contains $\mathcal{S}$ must also contain $\mathcal{B}$.

Prop: If Y has a topology generated by a subbasis $\mathcal{S}$

then $f: X \to Y$ is continuous

iff $f^{-1}(S)$ is open in X for all $S \in \mathcal{S}$

"subbasic open set"

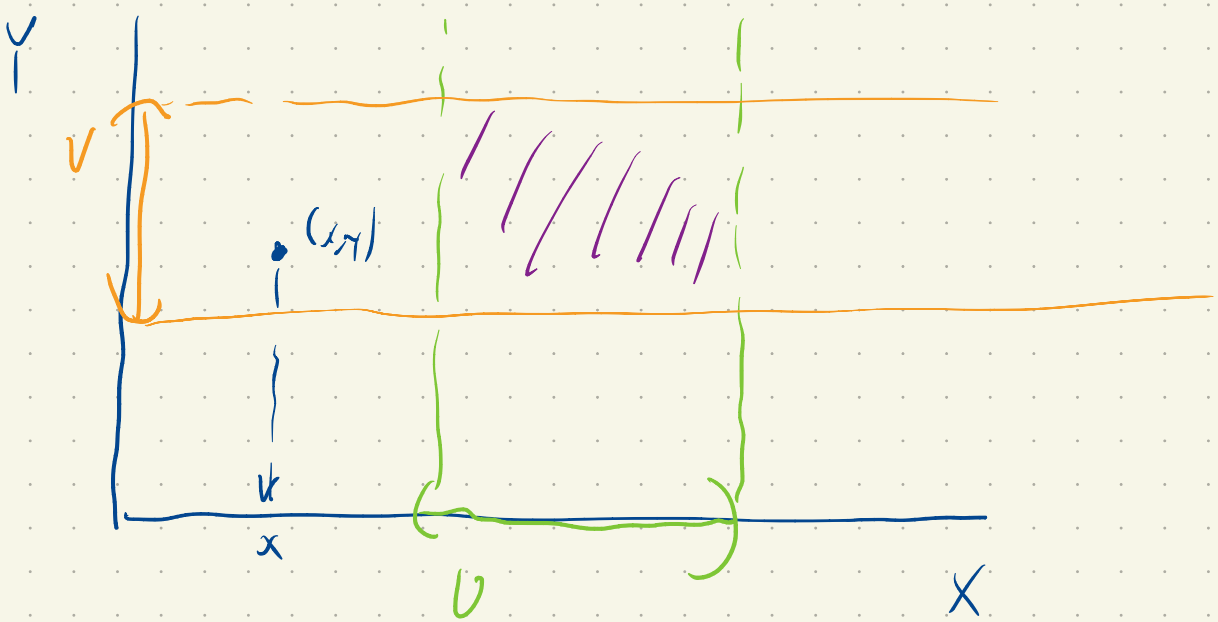
$$f^{-1}(S_1 \cap \dots \cap S_k) = \underbrace{\bigcap_{j=1}^{k} \underbrace{f^{-1}(S_j)}}$$

$X \times Y$

$$\mathcal{S} = \{\pi_X^{-1}(U) : \underset{\text{top on } X}{U \in \tau_X}\} \cup \{\pi_Y^{-1}(V) : V \in \tau_Y\}$$

$$\mathcal{S} \to \mathcal{B} \longrightarrow \tau_{X \times Y}$$

$\hookrightarrow$ product topology on $X \times Y$



New basic open sets have the form $\pi_X^{-1}(U) \cap \pi_Y^{-1}(V)$

U open in X

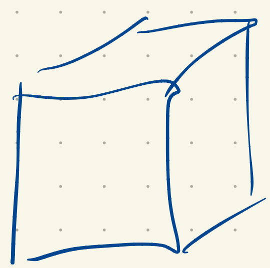
V open in Y

How about finitely many spaces $X_1, \dots, X_n$

$$\prod_{k=1}^{n} X_k = X_1 \times \cdots \times X_n$$

$$(x_1, \dots, x_n) \quad x_i \in X_i$$

Subbasis: $\mathcal{S} = \bigcup_{j=1}^{n} \{ \pi_j^{-1}(U) : U \subseteq X_j \text{ is open} \}$



$\mathbb{R}^n$ sadly now has two topologies.

$\hookrightarrow \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$

Are those the same?

$$\begin{tikzcd} & X \\ Z \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & A \arrow[u, hook, "i_A"'] \end{tikzcd}$$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[dr, "\tilde{f}_j"'] & \prod_{j=1}^{k} X_j \arrow[d, "\pi_j"] \\
& X_j
\end{tikzcd}$$

Characteristic Property of Product Topology:

$f : Z \to \prod_{j=1}^{k} X_j$ is continuous iff each $\tilde{f}_j = \pi_j \circ f$ is continuous.

Pf: Suppose f is continuous. The projections π_j are continuous by construction and hence so is each $\pi_j \circ f = \tilde{f}_j$.

Conversely, suppose each $\tilde{f}_j$ is continuous.

Consider a subbasic open set $\pi_j^{-1}(U) \subseteq \prod X_k$.

Then $f^{-1}(\pi_j^{-1}(U)) = (\pi_j \circ f)^{-1}(U)$

$$= \tilde{f}_j^{-1}(U)$$

which is open in Z as each $\tilde{f}_j$ is continuous,

Since $f^{-1}(S)$ is open for every subbasic open set S, f is continuous, $\blacksquare$

$f_j = \pi_j \circ f$

a component function

$\mathbb{R} \to \mathbb{R}^2$

$t \xmapsto{f} (\cos(t), \sin(t))$

π_1, π_2

$\pi_1 \circ f(t) = \cos(t)$

$\pi_2 \circ f(t) = \sin(t)$

A function $f: Z \to \prod X_k$ is continuous iff its component functions are continuous.

You are proving that $\mathbb{R}^n$ satisfies the char. property of the product top.

This begs the question: Is the c.p.p.t characteristic?

$$X = \prod_{k=1}^{n} X_k$$

$X_{(p)}$ product top

$X_{(r)}$ random top satisfying CPPT

Claim: $X_{(r)} = X_{(p)}$

We'll show this by showing

$$X_{(r)} \xrightarrow{Id_{rp}} X_{(p)}$$

$$X_{(p)} \xrightarrow{Id_{pr}} X_{(r)}$$

are continuous.

I claim $\pi_j^{(r)} : X_{(r)} \to X_j$ is continuous for each j.

$$\begin{tikzcd} X_{(r)} \arrow[r, "id_{(r)}"] \arrow[dr, "\pi_j^{(r)}"'] & X_{(r)} \arrow[d, "\pi_j^{(r)}"] \\ & X_j \end{tikzcd}$$

Since $id_{(r)} : X_{(r)} \to X_{(r)}$ is cts and since $X_{(r)}$ satisfies CAPT, $\pi_j^{(r)}$ is continuous.

$$
\begin{tikzcd}
X_{(r)} \arrow[r, "id_{rp}"] \arrow[dr, "\pi_j^{(r)}"'] & X_{(p)} \arrow[d, "\pi_j^{(p)}"] \\
& X_j
\end{tikzcd}
$$

Since $\pi_j^{(r)}$ is cts, so is id_{rp}.

for each j.

Facts: 1) A (finite) product of Hausdorff spaces is Hausdorff.

Exercise.

2) If $\underline{\mathcal{B}_1}$ is a basis for X_1 and $\underline{\mathcal{B}_2}$ is a basis for X_2, $\{B_1 \times B_2 : B_1 \in \mathcal{B}_1 \text{ and } B_2 \in \mathcal{B}_2\}$ is a basis for $X_1 \times X_2$.

HW