$$\begin{tikzcd} & X \\ Z \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & A \arrow[u, hook, "i_A"'] \end{tikzcd}$$

f is cts $\Longleftrightarrow$ $\tilde{f}$ is cts.

We will say that a topology on A satisfies the char property of the subspace topology if whenever

$f : Z \to A$ is a map then f is cts iff $i_A \circ f$ is cts.

"The characteristic property is characteristic"

Let A_s be A with the subspace topology.

Let A_r be A with some random topology satisfying the char. property.

Want to show $A_s = A_r$.

$$(A, \tau_s) \qquad (A, \tau_r)$$

$$Id_{s,r} : A_s \to A_r$$

$$Id_{r,s} : A_r \to A_s \qquad Id_{r,s} = Id_{s,r}^{-1}$$

$A_s = A_r$ if these two maps are continuous.

I

$$\begin{tikzcd} & X \\ A_r \arrow[ur, "\widetilde{\mathrm{Id}}_{r,r} \mapsto i_{A_r}"] \arrow[r, "\mathrm{Id}_{r,r}"'] & A_r \arrow[u, hook, "i_{A_r}"'] \end{tikzcd}$$

II

$$\begin{tikzcd} & X \\ A_r \arrow[ur, hook, "i_{A_r}"] \arrow[r, "\mathrm{Id}_{r,s}"'] & A_s \arrow[u, hook, "i_{A_s}"'] \end{tikzcd}$$

III

$$\begin{tikzcd} & X \\ A_s \arrow[ur, hook, "i_{A_s}"] \arrow[r, "\mathrm{Id}_{s,r}"'] & A_r \arrow[u, hook, "i_{A_r}"'] \end{tikzcd}$$

From I, since $\mathrm{Id}_{r,r}$ is cts, since A_r satisfies the CPST, i_{A_r} is continuous. Because A_s satisfies the CPST diagram II and the continuity of i_{A_r}, $\mathrm{Id}_{r,s}$ is continuous.

From diagram III an analogous argument shows $\mathrm{Id}_{s,r}$ is cts.

Prop: Suppose X is a top space and

$$A \subseteq B \subseteq X.$$

Then the subspace topologies on A as a subset of B or X coincide.

Pf: Let A_B and A_X denote the two subspace topologies. We'll show that $A_B = A_X$ by showing A_B satisfies the char property of the subspace topology w.r.t. X.

Let $f: Z \to A_B$ be a map, and consider

$$\begin{tikzcd}
 & X \\
 & B \arrow[u, hook, "i_B"'] \\
Z \arrow[r, "f"] \arrow[uur, "\tilde{f}"] \arrow[ur, "\hat{f}"] & A_B \arrow[u, hook, "i_{A_B}"']
\end{tikzcd}$$

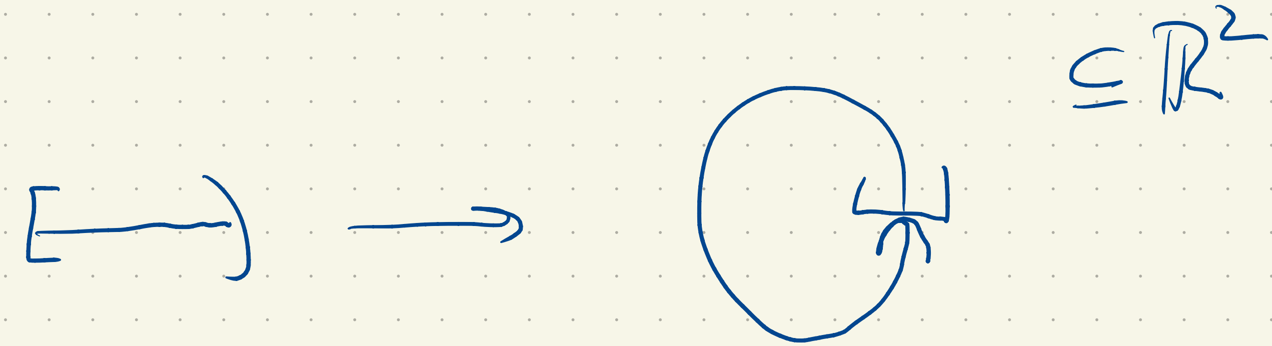
Suppose f is continuous. Then $\tilde{f} = i_B \circ i_{A_B} \circ f$ is a composition of continuous functions and is cts.

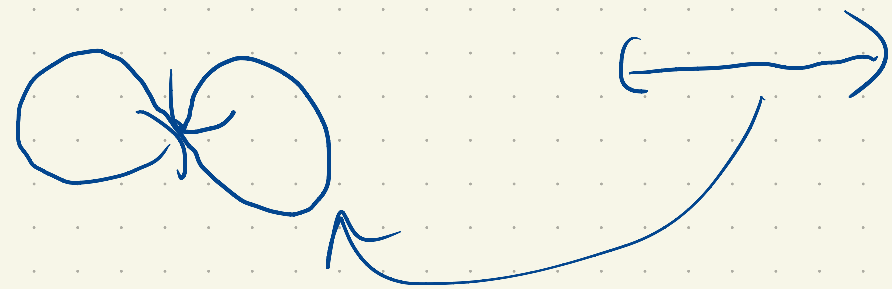
Conversely, suppose $\tilde{f}$ is cts. From the CPST on B we conclude $\hat{f}$ is cts and from the CPST applied to A_B we find f is cts.

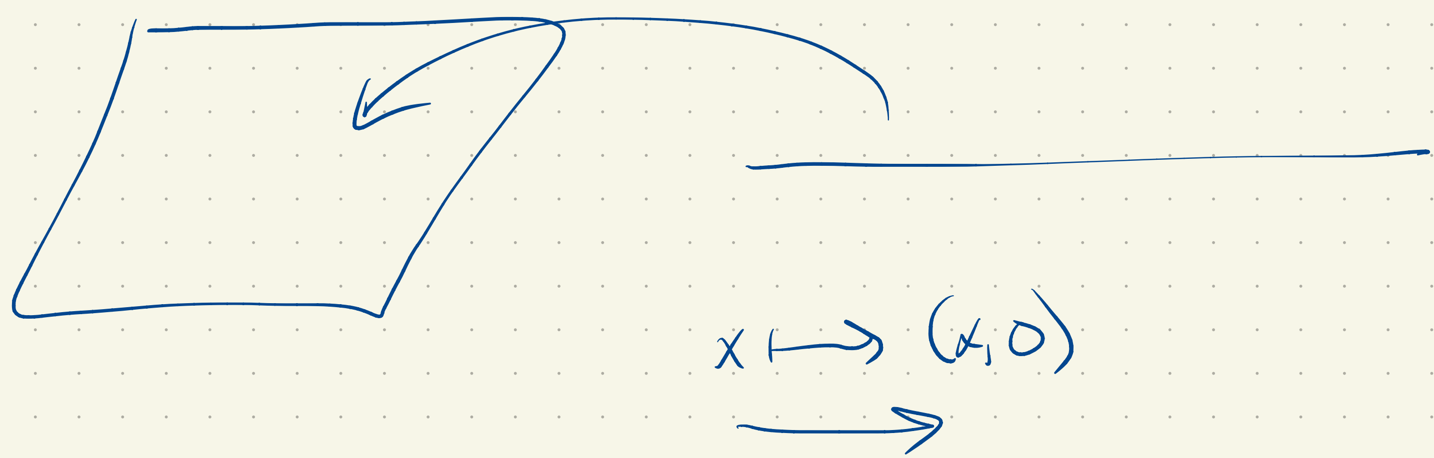
Def: A map $f: X \to Y$ is a topological embedding if f is a homeomorphism onto $f(X)$ (with the subspace top.)

Necessary:
1) f is continuous
2) f is injective

(surjectivity is free!)

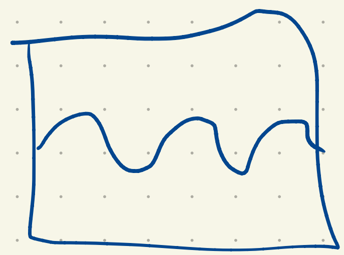

$\subseteq \mathbb{R}^2$




$x \longmapsto (x,0)$

$\pi : \mathbb{R}^2 \to \mathbb{R}$

$\pi(x,y) = x$



$U \subseteq \mathbb{R}^n$

$f : U \to \mathbb{R}^k$, continuous

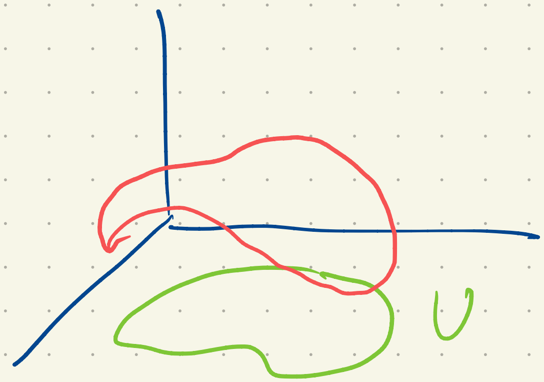
Graph of f $\quad \Gamma_f = \{ (x, f(x)) \in \mathbb{R}^{n+k} : x \in U \}$

$\Phi : U \to \mathbb{R}^{n+k}$ is a top. embedding.

$\Phi(x) = (x, f(x))$

Φ is clearly injective and continuous.

Its inverse is just projection restricted to Γ_f and is cts.



$f: U \to \mathbb{R}$

S^2_+

S^2 is a manifold.

Hausdorff, 2^{nd} countable are free (from $\mathbb{R}^3$)

$S^2_+ = \{ x \in S^2 : x_3 > 0 \}$

S^2_+ is an open subset of S^2:

a) $(0, \infty)$ is open in $\mathbb{R}$.

b) $\pi_3 : \mathbb{R}^3 \to \mathbb{R}$ $\pi_3(x_1, x_2, x_3) = x_3$ is continuous.

c) $\pi_3^{-1}((0,\infty))$ is open in $\mathbb{R}^3$

d) $S^2_+ = S^2 \cap \pi_3^{-1}((0,\infty))$.

We saw earlier that S^2_+ is homeomorphic to an open subset of $\mathbb{R}^2$.

(as a subspace of $\mathbb{R}^3$)

But S^2_+ has the same topology as a subspace of S^2.

S^2_+ is an open set in S^2 that is homeomorphic to an open set in $\mathbb{R}^2$.

Now consider $\Phi : S^2 \to S^2$

$$\Phi(x,y,z) = (x,y,-z)$$

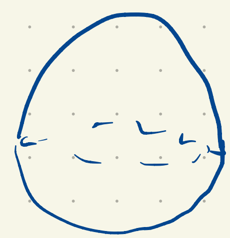
This is continuous ($\mathbb{R}^3 \to \mathbb{R}^3$ and then by restriction $S^2 \to S^2$)

and is its own inverse. It's a homeomorphism,

$$\Phi(S^2_+) = S^2_- = \{(x,y,z)\in S^2 : z<0\}.$$

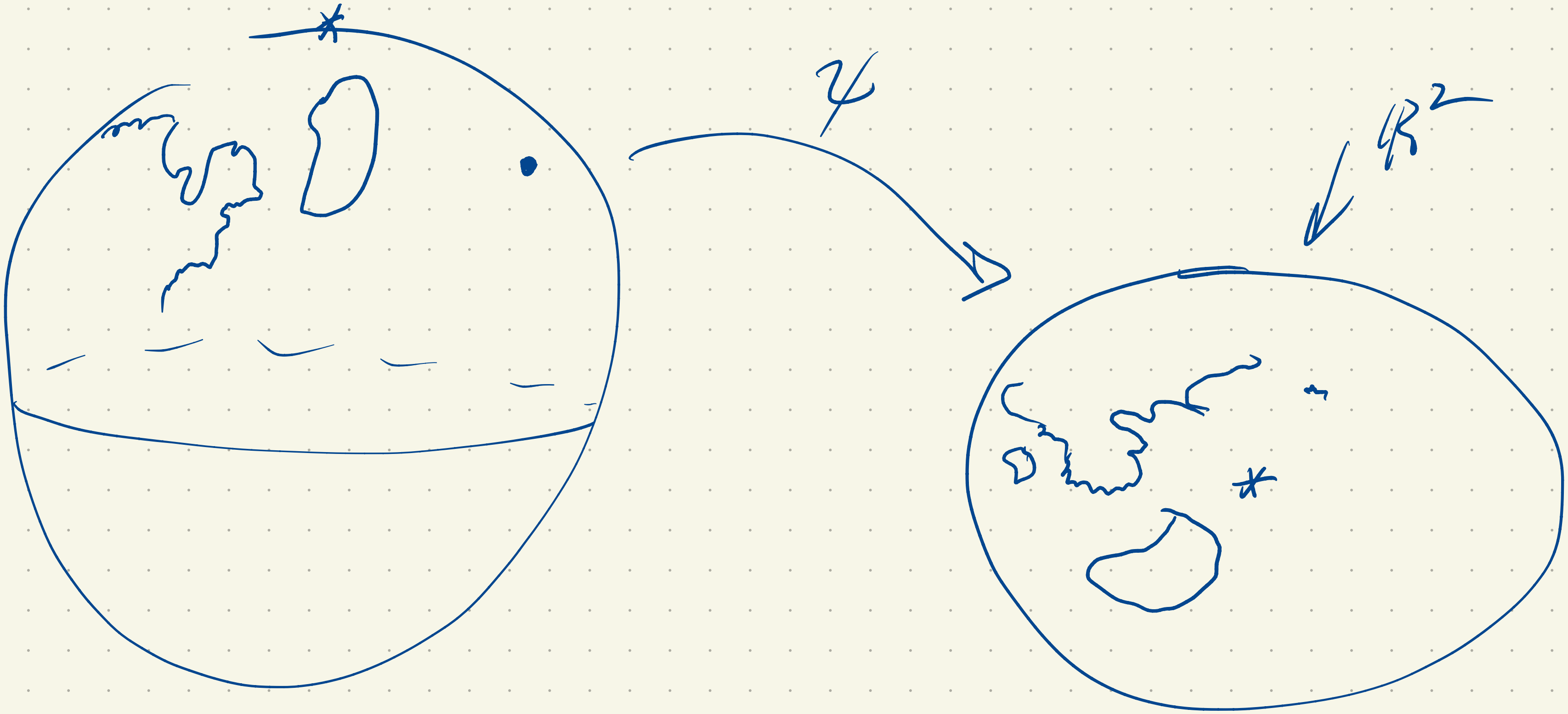
$$\Phi\big|_{S^2_+} : S^2_+ \to S^2_-$$ is a homeomorphism and S^2_- is open in S^2.

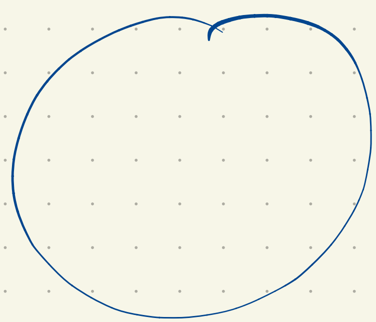
Consequently S^2_- is an open set in S^2 and homeomorphic to an open set in $\mathbb{R}^2$.

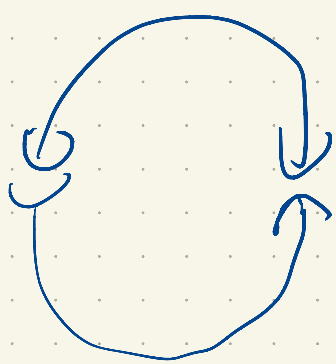


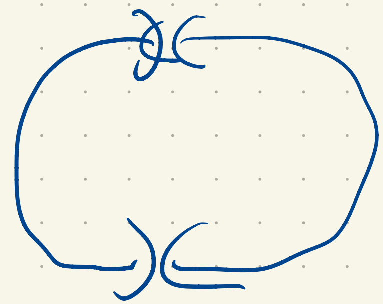
$$\Phi(x,y,z) = (z,y,x)$$ is a homeo $S^2 \to S^2$.

$x>0$


ψ
$\mathbb{R}^2$

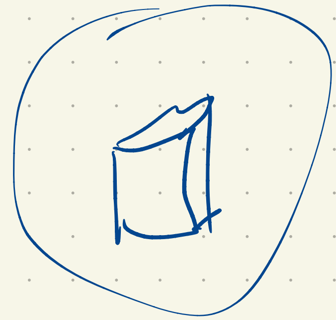






$$X \longrightarrow \mathbb{R}^n$$

metric



$$f(x) = (f_1(x), \dots, f_n(x))$$

$x_k \to x$ in X

$f(x_k) \to f(x)$

$\{z_k\}$ $\quad z_k \in \mathbb{R}^n$

$z_k \to z$

$z_{k,i} \to z_i$

$\{z_k\}$ in $\mathbb{R}^n$ converges iff each $\pi_i(z_k)$ converges.