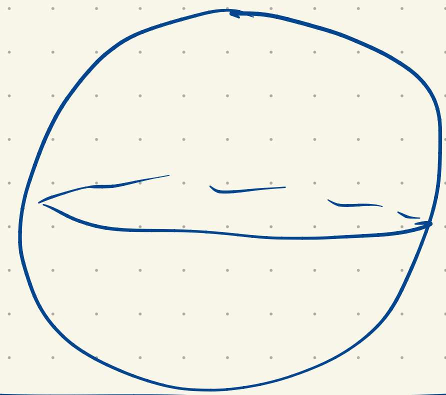
The components of ψ are called coordinates

$$\psi(p) = (\psi_1(p), \psi_2(p))$$



Chapter 3 New spaces from old.

Given a subset $A \subseteq X$ we'll put a natural topology on A.

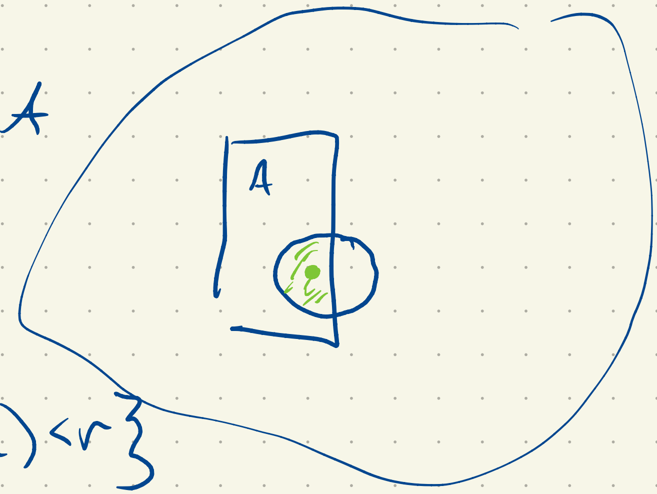
We've already seen that if $U \subseteq X$ is open, it has a topology consisting of the open sets in X contained in U.

If X is a metric space and $A \subseteq X$ then A inherits a metric and has a metric space topology.

A set W in a metric space is open if

for all $x \in W$ $\exists$ $r > 0$ s.t. $B_r(x) \subseteq W$.

If $A \subseteq X \leftarrow$ metric space and $a \in A$



$$B_r^A(a) = B_r^X(a) \cap A$$

$$\downarrow \qquad = \{ x \in X : x \in A, d(x,a) < r \}$$

$$\{ x \in A : d(x,a) < r \}$$

Suppose $U \subseteq A$ is open. Then

$$U = \bigcup_{a \in A} B_{r_a}^A(a) = \bigcup_{a \in A} \left(B_{r_a}^X(a) \cap A \right)$$

$$= \left[\bigcup_{a \in A} B_{r_a}^X(a) \right] \cap A$$

$$= \underbrace{\hat{U}}_{\text{open in } X} \cap A$$

Every open set in A is the intersection of an open set in X with A.

Conversely suppose $\hat{U} \subseteq X$ is open and let $U = \hat{U} \cap A$.

If $a \in U$ then $a \in \hat{U}$ so there exists $r > 0$ with

$B_r^X(a) \subseteq \hat{U}$. But then $a \in \underbrace{B_r^X(a) \cap A}_{B_r^A(a)} \subseteq \hat{U} \cap A = U$.

$$a \in B_r^A(a) \subseteq U.$$

So U is open in A.

Open sets in A are precisely open sets in X intersected with A.

Def: Let X be a top space and let $A \subseteq X$.

The <u>subspace topology</u> on A is

$$\tau_A = \{ U \cap A : U \text{ is open in } X \}.$$

Exercise: τ_A is a topology.

Properties of the ambient space X are often inherited by $A \subseteq X$.

Prop: Suppose X is Hausdorff. Then $A \subseteq X$ is also Hausdorff.

Pf: Let $a_1, a_2 \in A$ with $a_1 \neq a_2$.

Since X is Hausdorff we can find open sets $\hat{U}_1, \hat{U}_2$ in X such that $a_i \in \hat{U}_i$ and $\hat{U}_1 \cap \hat{U}_2 = \emptyset$.

Let $U_i = \hat{U}_i \cap A$ so U_i is open in A.

Clearly each $a_i \in U_i$ and the sets U_i are disjoint.

Prop: If $\hat{\mathcal{B}}$ is a basis for X and if $A \subseteq X$ then

$$\mathcal{B} = \{ \hat{B} \cap A : \hat{B} \in \hat{\mathcal{B}} \}$$

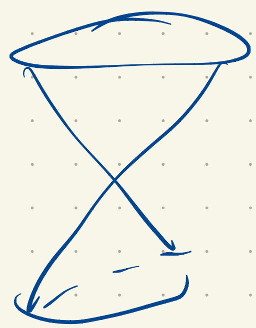
is a basis for the subspace topology on A.

Pf: Observe, by the definition of the subspace topology and the fact that elements of $\hat{\mathcal{B}}$ are open in X, each $B \in \mathcal{B}$ is open in A. Moreover, suppose $U \subseteq A$ is open and $a \in U$. There exists $\hat{U}$, open in X, such that $U = \hat{U} \cap A$. Since $\hat{\mathcal{B}}$ is a basis for X there exists $\hat{B} \in \hat{\mathcal{B}}$ such that $a \in \hat{B} \subseteq \hat{U}$. But then $a \in \hat{B} \cap A \subseteq \hat{U} \cap A = U$. Since $\hat{B} \cap A \in \mathcal{B}$, $\mathcal{B}$ is a basis.

Cor: If X is second countable and $A \subseteq X$ then A is second countable.

Exercise: If X is first countable and $A \subseteq X$ then A is 1^{st} countable.

$\mathbb{R}^3$



Note: If $A \subseteq X$ and X is a metric space then the subspace top on A is precisely the metric space top. on A.

$$A \subseteq B \subseteq X$$

has two topologies!

$A \subseteq B$ so A has the subspace top as a subspace of B

$$\tau_A^B$$

$A \subseteq X$, τ_A^X Are these the same?

"Characteristic Property of Subspace Topology"

$$A \subseteq X$$

$$A \xhookrightarrow{i_A} X$$

$$i_A(a) = a$$

Is i_A continuous?

Suppose $\hat{U}$ is open in X,

$$i_A^{-1}(\hat{U}) = \{a \in A : i_A(a) \in \hat{U}\}$$

$$= \{a \in A : a \in \hat{U}\}$$

$$= \hat{U} \cap A \text{ which is open in } A.$$

Yes, i_A is continuous.

In fact, if A has a topology τ and $i_A : A \to X$ is continuous with respect to τ then $\tau_A \subseteq \tau$.

subspace

The subspace topology is the coarsest topology on A such that i_A is continuous.

Consider a function $f : Z \to A$

$\uparrow$

other top space.

$$
\begin{tikzcd}
 & X \\
Z \arrow[r, "f"'] \arrow[ur, dashed, "\hat{f}"] & A \arrow[u, hook, "i_A"']
\end{tikzcd}
\qquad\qquad
\hat{f} = i_A \circ f
$$

If f is continuous then $\hat{f}$ is a composition of continuous functions and is hence continuous.

What if $\hat{f}$ is continuous? Is f continuous?

Let $U \subseteq A$ be open in A. Then $U = \hat{U} \cap A$ for some open set U in X.

But then

$$
\begin{aligned}
f^{-1}(U) &= f^{-1}(\hat{U} \cap A) \\
&= f^{-1}(i_A^{-1}(\hat{U})) \\
&= (i_A \circ f)^{-1}(\hat{U}) \\
&= \hat{f}^{-1}(\hat{U})
\end{aligned}
$$

which is open in Z. Hence f is continuous.

Prop (Char. Property of Subspace Topology)

If X is a top. space and $A \subseteq X$ then a map

$f : Z \to A$ for some top. space Z is continuous

iff $i_A \circ f$ is continuous.

$$\begin{tikzcd} & X \\ Z \arrow[r, "f"'] \arrow[ur, "i_A \circ f"] & A \arrow[u, hook, "i_A"'] \end{tikzcd}$$

Map into A are continuous iff they are continuous if thought of as continuous into the ambient space.

Consequence: we can restrict codomain without affecting continuity.

If $f: Z \to X$ is continuous and $f(Z) \subseteq A \subseteq X$ then $\tilde{f}: Z \to A$ is continuous.

On the other hand, we can also restrict domains and preserve continuity.

Prop: If $f: X \to Y$ is continuous and $A \subseteq X$ then

$$f|_A : A \to Y \quad \text{is continuous.}$$

Pf: $$f|_A = f \circ i_A.$$

$$A \subseteq B \subseteq X$$

Is $\tau_A^B = \tau_A^X$?

$$id^{BX} : (A, \tau_A^B) \to (A, \tau_A^X), \quad id^{BX}(a) = a$$

$$id^{XB} : (A, \tau_A^X) \to (A, \tau_A^B), \quad id^{XB}(a) = a$$

Clearly id^{BX} is a bijection with inverse id^{XB}.

To show $\tau_A^B = \tau_A^X$ it suffices to show id^{BX} and id^{XB} are continuous.