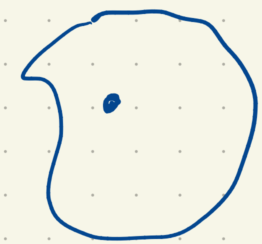
First countability:

$\forall p \in X$ there is a countable collection $\{U_k\}$ of open sets containing p such that ~~for~~ all open sets U containing p there exists k with

$$p \in U_k \subseteq U.$$



First countable spaces: $\mathbb{R}$ $\quad x \in \mathbb{R}$ $\quad (x-\frac{1}{n}, x+\frac{1}{n})$

metric spaces $\quad U_n = B_{1/n}(p)$

X, discrete.

Def: A nested nbhd basis at $p \in X$ is a countable nbhd basis at p $\{W_k\}$ with

$W_{k+1} \subseteq W_k$ for all k. $\left(k_1 \geq k_2 \Rightarrow W_{k_1} \subseteq W_{k_2} \right)$

Lemma: A point $p \in X$ admits a countable nbhd basis iff it admits a nested nbhd basis

Pf: (of non obvious direction)

Suppose $\{W_k\}$ is a countable nbhd basis at p.

For each k define $\hat{W}_k = \bigcap_{j=1}^{k} W_k$. I claim that $\{\hat{W}_k\}$ is a nested nbhd basis. Clearly this is a countable collection of nested open sets and hence it suffices to show that it is a nbhd basis at p. Let U be an open set contains p. There exists some k with $p \in W_k \subseteq U$. But $\hat{W}_k \subseteq W_k$ so $p \in \hat{W}_k \subseteq W_k \subseteq U$. $\square$

In first countable spaces we can frequently argue about closures using sequences.

Lemma 2.48 in the text has several flavors of this.

Lemma: Let X be first countable and let $A \subseteq X$. Then $p \in \overline{A}$ iff there exists a sequence in A converging to p.

Pf: If a sequence in A converges to p then p is a contact point of A and hence $p \in \overline{A}$.

Conversely, suppose $p \in \bar{A}$ and is hence a contact point of A. Let $\{W_k\}$ be a nested nbhd base at p. Since p is a contact point of A, for each k we can find $a_k \in W_k \cap A$. To see that $a_k \to p$ let U be an open set containing p. There exists a W_K with $p \in W_K \subseteq U$. If $k \geq K$ then

$$a_k \in W_k \subseteq W_K \subseteq U. \qquad \square$$

It's hard to find examples of non first countable spaces. (HW)

Def: A space is $\underline{\text{2}^{\text{nd}}\text{ countable}}$ if it admits a countable basis.

Observations

1) 2^{nd} countable $\Rightarrow$ first countable.

2) The discrete topology on $\mathbb{R}$ is not 2^{nd} countable.

$$x \in B \subseteq \{x\}$$

3) $\mathbb{R}$ is 2^{nd} countable.

(a,b) $a<b$, $a,b \in \mathbb{Q}$.

4) $\mathbb{R}^2$ is 2^{nd} countable.

Balls with rational radii and centers with rational coords.

Def. A topological space is <u>separable</u> if it admits a countable dense subset.

2^{nd} countable $\Rightarrow$ separable

$\{W_k\}$ $\quad p_k \in W_k$ $\quad$ Now show $\{p_k\}$ is dense.

(A is dense in X if $\overline{A} = X$

i.e., each x in X is a contact point of A)

Def: Let X be a top space. A collection $\{U_\alpha\}_{\alpha \in I}$ of open sets is an <u>open cover</u> of X if $\displaystyle\bigcup_{\alpha \in I} U_\alpha = X$.

Def: A topological space X is Lindelöff if every open cover of X admits a countable subcover. ["just short" of compact]

Proposition: Every 2^{nd} countable space is Lindelöff.

Pf: Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of the 2^{nd} countable space X. Let $\{W_k\}$ be a countable basis for X.

Let $\mathcal{H}$ be the set of indices k such that $W_k \subseteq U_\alpha$ for some α.

For each $k \in \mathcal{H}$ we can then pick α_k with $W_k \subseteq U_{\alpha_k}$.

I claim that $\{U_{\alpha_k}\}_{k\in\mathcal{K}}$ covers X.

Let $p\in X$ and pick some U_α with $p\in U_\alpha$.

Since $\{W_k\}$ are a basis there exists W_k with

$$p\in W_k\subseteq U_\alpha.$$ Observe that $k\in\mathcal{K}$.

Hence $p\in W_k\subseteq U_{\alpha_k}$ for some $k\in\mathcal{K}$.

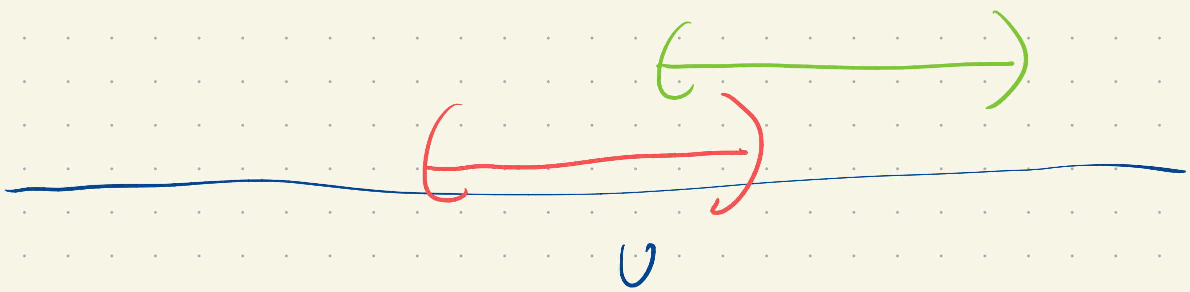
$$2^{nd}\text{ countable} \Rightarrow \left\{\begin{array}{l}1^{st}\text{ countable}\\ \text{separable}\\ \text{Lindelöff}\end{array}\right]$$

all of these are independent and no combination leads to a reverse implication.

Prop: If X is 2^{nd} countable and U is open in X then U is 2^{nd} countable.

Pf sketch: Let $\mathcal{B}$ be a countable basis for X.

Let $\mathcal{B}' = \{ B \in \mathcal{B} ; \ B \subseteq U \}$



Manifolds

A manifold is a top. space that "looks like" $\mathbb{R}^n$ locally with the same n at every point.

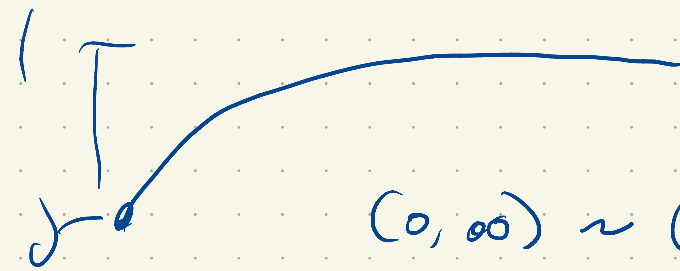
Def: A space X is **locally Euclidean** **with dimension** $n \in \mathbb{N}$

if each $p \in X$ admits a neighborhood U

that is homeomorphic to $\mathbb{R}^n$.

Alternatives

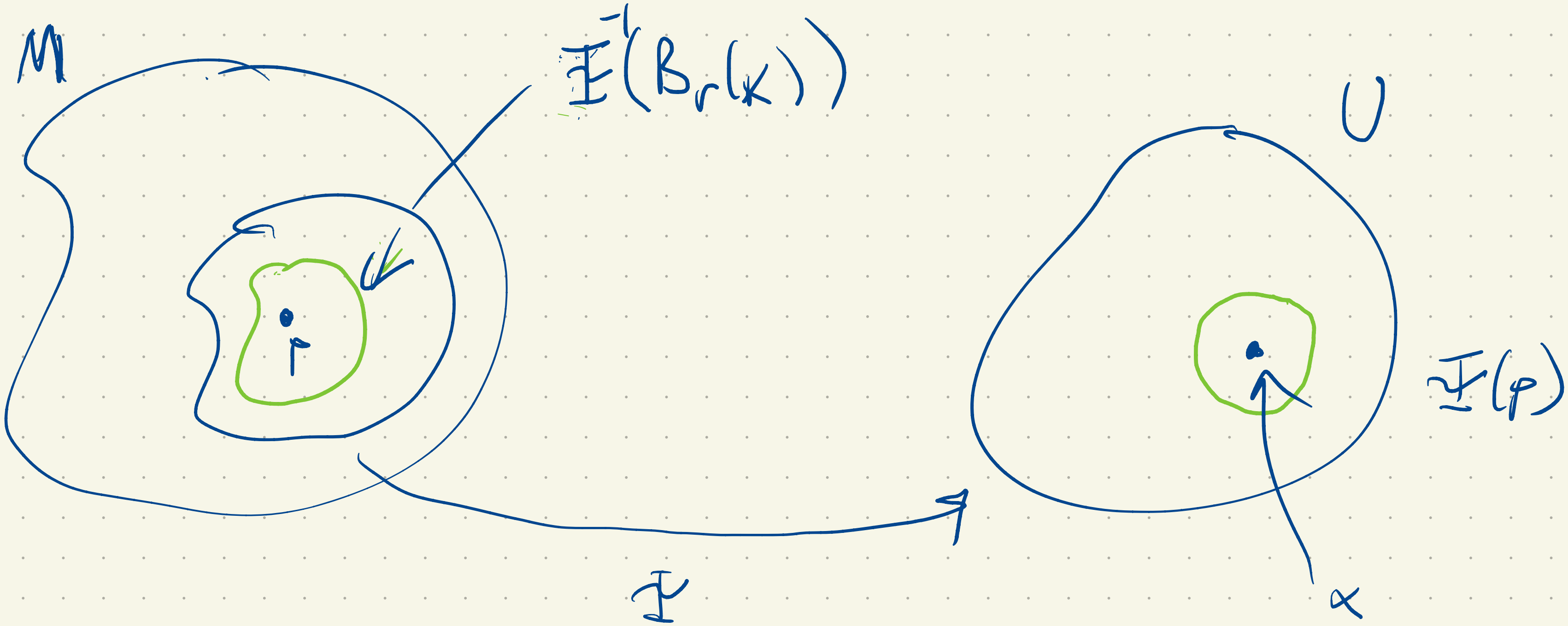
- $B_1(0) \subseteq \mathbb{R}^n$



$(0, \infty) \sim (0,1)$

- $B_r(0) \subseteq \mathbb{R}^n$ for some r

- $U \subseteq \mathbb{R}^n$ where U is open.



Claim: $\Phi\big|_{\Phi^{-1}(B_r(x))} : \Phi^{-1}(B_r(x)) \to B_r(x)$ is a homeomorphism