Def: Let (X, τ) be a topological space.

A <u>basis</u> for the topology is a collection $\mathcal{B} \subseteq \tau$ such that for all $U \in \tau$ there exists a subcollection $\mathcal{B}' \subseteq \mathcal{B}$ with

$$U = \bigcup_{B \in \mathcal{B}'} B.$$

Note: to show some collection $\mathcal{B}$ of subsets of X is a basis for the topology you need to:

1) Show that the sets in $\mathcal{B}$ are open

2) Every open set is a union of things in $\mathcal{B}$.

1) is easy to forget.

Exercise: 2) is the same as

"for all $U \in \tau$ and all $p \in U$ there exists $B \in \mathcal{B}$ with $p \in B \subseteq U$"

Bases need not be unique.

Frequently if you need to show something is true about every open set, it is enough to show the same fact is true about every set in some basis.

Prop: Suppose $f: X \to Y$, and that $\mathcal{B}$ is a basis for the topology on Y. Then f is continuous iff $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}$.

Pf: Suppose f is continuous. Since each $B \in \mathcal{B}$ is open in Y, $f^{-1}(B)$ is open in X.

Conversely suppose $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}$.

Let U be open in Y. Then $U = \bigcup_{\alpha \in I} B_\alpha$ for some collection $\{B_\alpha\}_{\alpha \in I} \subseteq \mathcal{B}$. But then

$$f^{-1}(U) = f^{-1}\left(\bigcup_{\alpha \in I} B_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(B_\alpha).$$

Since each $f^{-1}(B_\alpha)$ is open, so is U.

Def: Let X be a set.

A prebasis in X is a collection $\mathcal{B}$ of subsets of X such that

1) $\bigcup \mathcal{B} = \bigcup_{B \in \mathcal{B}} B = X$

2) For all $B_1, B_2 \in \mathcal{B}$ and all $p \in B_1 \cap B_2$,

there exists $B_3 \in \mathcal{B}$ with

$$p \in B_3 \subseteq B_1 \cap B_2.$$

[Every $B_1 \cap B_2$ is a union of elements of $\mathcal{B}$]

Given a prebasis $\mathcal{B}$ define

$$\tau_{\mathcal{B}} = \{U \subseteq X : U \text{ is a union of elements of } \mathcal{B}\}.$$

$$= \{U \subseteq X : \text{for all } p \in U \text{ there exists } B \in \mathcal{B} \text{ with } p \in B \subseteq U\}$$

Claim: $\tau_{\mathcal{B}}$ is a topology and $\mathcal{B}$ is a basis for this topology.

Is $\emptyset \in \tau_{\mathcal{B}}$? Yep, trivial.

Is $X \in \tau_{\mathcal{B}}$? Yes, by definition

Given $\{U_\alpha\}_{\alpha \in I} \subseteq \tau_{\mathcal{B}}$ consider some $p \in \bigcup_{\alpha \in I} U_\alpha$.

Then there exists $\alpha \in I$ with $p \in U_\alpha$ and hence

some $B \in \mathcal{B}$ with $p \in B \subseteq U_\alpha \subseteq \bigcup_{\beta \in I} U_\beta$.

So $\bigcup_{\beta \in I} U_\beta \in \tau_{\mathcal{B}}$.

Suppose $U_1, U_2 \in \tau_{\mathcal{B}}$. Let $p \in U_1 \cap U_2$.

There exist $B_1, B_2 \in \mathcal{B}$ with $p \in B_i \subseteq U_i$.

But there exists $B_3 \in \mathcal{B}$ with

$p \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. So $U_1 \cap U_2 \in \tau_{\mathcal{B}}$.

The same holds for any finite intersection by induction.

Homework: $\tau_{\mathcal{B}}$ is the smallest topology for which each $B \in \mathcal{B}$ is open.

Is $\mathcal{B}$ a basis for $\tau_{\mathcal{B}}$?

Is each $B \in \mathcal{B}$ also in $\tau_{\mathcal{B}}$?. Trivial.

Moreover each $U \in \tau_{\mathcal{B}}$ is a union of elements of $\mathcal{B}$ by construction.

Yep!

Given a top. space (X, τ) and a basis $\mathcal{B}$ for the topology, in fact $\mathcal{B}$ is a prebasis in X and generates τ. exercise.

Examples. 1) $\mathbb{R}$, $\mathcal{B} = \{ (a,b) : a < b \}$

Prebasis $\mathbb{R} = \bigcup_{x \in \mathbb{R}} (x-1, x+1)$

An intersection of open intervals is either empty or an element of $\mathcal{B}$ itself.

What is $\tau_{\mathcal{B}}$? Pick $U \in \tau_{\mathbb{R}}$. By def given $p \in U$ there is an interval (a,b) with

Each $B \in \mathcal{B}$ is open in the standard top.

$p \in (a,b) \subseteq U$.

$\mathcal{B}$ is a basis for the topology on $\mathbb{R}$.

$$\tau_{\mathcal{B}} = \tau_{\mathbb{R}} ?$$

$$\tau_{\mathcal{B}} \subseteq \tau_{\mathbb{R}}$$

Conversely, given $U \in \tau_{\mathbb{R}}$ we just argued that $U \in \tau_{\mathcal{B}}$. So $\tau_{\mathbb{R}} \subseteq \tau_{\mathcal{B}}$.

Exercise: If $\mathcal{B}$ is a basis for topologies τ_1, τ_2 on X then $\tau_1 = \tau_2$.

b) $\mathbb{R}$, $\mathcal{B} = \{ (a,b) \subseteq \mathbb{R} : a,b \in \mathbb{Q} \}$

This is a prebasis by the same argument (almost) as above.

$\tau_{\mathcal{B}} = \tau_{\mathbb{R}}$

$\tau_{\mathcal{B}} \subseteq \tau_{\mathbb{R}}$ since each $B \in \mathcal{B}$ is open in $\tau_{\mathbb{R}}$.

$\tau_{\mathbb{R}} \subseteq \tau_{\mathcal{B}}$ since each interval $(\hat{a},\hat{b})$ $\hat{a},\hat{b} \in \mathbb{R}$ is in $\tau_{\mathcal{B}}$.

c) X any set $\mathcal{B} = \{ \{x\} : x \in X \}$

Prebasis? Yep

$\tau_{\mathcal{B}} = \tau_{disc.}$

$\tau_{\mathcal{B}} \subseteq \tau_{disc}$

Each $U \in \tau_{disc}$ is a union of elements from $\tau_{\mathcal{B}}$.

So $\tau_{disc} \subseteq \tau_{\mathcal{B}}$.

d) X any set $\mathcal{B} = \{X\}$.

$\tau_{\mathcal{B}} = \tau_{ind}$ (exercise)

e) $\mathbb{R}$ $\mathcal{B} = \{ [a,b) : a < b \}$.

Prebasis? $\mathbb{R} = \bigcup_{x \in \mathbb{R}} [x, x+1)$

The intersection of $B_1, B_2 \in \mathcal{B}$ is either empty or in $\mathcal{B}$.

We get a topology $\tau_{\mathcal{B}} = \tau_{\text{lower limit}}$.

Since $[0,1) \in \tau_{\mathcal{B}}$ but $[0,1) \notin \tau_{\mathbb{R}}$,

$$\tau_{\mathcal{B}} \not\subseteq \tau_{\mathbb{R}}.$$

Is the reverse inclusion true?

Each (a,b) is a union of sets $[a_n, b)$ with $a_n \to a$

So each $(a,b) \in \tau_{\mathcal{B}}$. So $\tau_{\mathbb{R}} \subseteq \tau_{\mathcal{B}}$.

It's strictly smaller.

More open sets makes it harder for sequences to converge.

$x_n = \frac{-1}{n}$ $\quad x_n \to 0$ in $\tau_{\mathbb{R}}$

$x_n \not\to 0$ in $\tau_{\mathcal{B}}$. ($[0,1)$ excludes all the x_n's.)

Next dest: Axioms "limiting richness"

Countability axioms.

Def: A neighborhood basis at $p \in X$ is a collection $\mathcal{W} \subseteq \mathcal{U}(p)$ such that for all $U \in \mathcal{U}(p)$, $\exists$ $W \in \mathcal{W}$ with $p \in W \subseteq U$.

"neighborhood base"

Def: A space is <u>first countable</u> if each $p \in X$ admits a countable neighborhood basis.