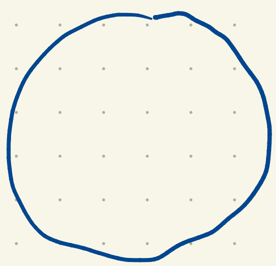
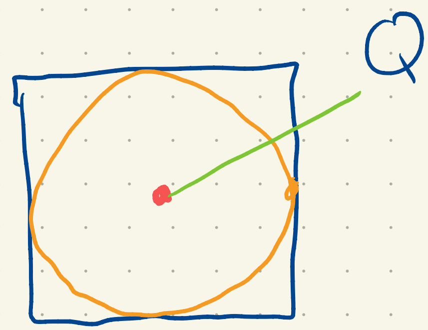
HW: cts, open, closed are all independent

Def: A map $f: X \to Y$ is a <u>homeomorphism</u> if it is a bijection, is continuous, and has a continuous inverse.

e.g.





$\{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$ $\{x \in \mathbb{R}^2 : \|x\|_\infty = 1\}$

Each a metric space inheriting a metric from $\mathbb{R}^2$.

$f: S^1 \to Q$ $g: Q \to S^1$

$$x \longmapsto \frac{x}{\|x\|_\infty} \qquad\qquad x \longmapsto \frac{x}{\|x\|_2}$$

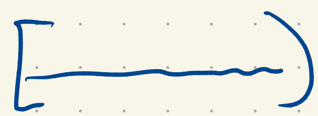
Tools needed to show f, g are cts rigorously

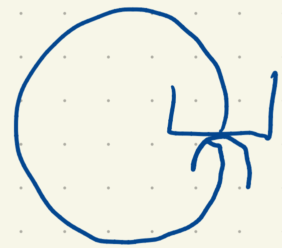
- $(x,y) \longrightarrow x$ is continuous
- $Z \longrightarrow \mathbb{R}^2$ is cts iff x and y are.
  $z \longmapsto (x(z), y(z))$
- compositions of cts functions are cts
- If $A \subseteq \mathbb{R}^2$ and $f: \mathbb{R}^2 \to Z$ is cts
  then $f|_A \to Z$ is cts.

Caution: if $f: X \to Y$ is a continuous bijection
f^{-1} need not be continuous.

$f: [0,1) \to S^1$

$f(t) = (\cos(2\pi t), \sin(2\pi t))$ $\qquad (f(t) = e^{2\pi i t})$





Obviously a continuous bijection

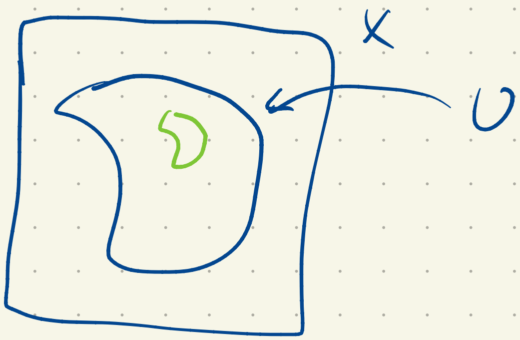
On HW: f^{-1} is not cts.

Just because this f is not a homeomorphism we don't know that no possible homeomorphism exists.

Technical Observation: "continuity is a local property"

Let (X, τ) be a topological space and let $U \in \tau$.

Then U inherits a topology: $\tau_U = \{ V \in \tau : V \subseteq U \}$



Exercise: this is a topology.

It's easy to see that $A \subseteq U$ is open in U if and only if A is open in X.

Given some $f: X \to Y$ we have the restriction of f to U

$$f|_U : U \to Y$$

$$f|_U(p) = f(p)$$

Exercise: If $f: X \to Y$ is continuous then $f|_U : U \to Y$ is also continuous.

$$f|_U^{-1}(W) = U \cap f^{-1}(W)$$

A kind of converse of this is true:

Prop: Suppose $f: X \to Y$ and for each $p \in X$ there exists $U_p \in \mathcal{U}(p)$ such that $f|_{U_p} : U_p \to Y$ is continuous.

Then f is continuous.

Pf: Observe that $X = \bigcup_{p \in X} U_p$. Let $W \subseteq Y$ be open.

Then

$$\begin{aligned} f^{-1}(W) &= X \cap f^{-1}(W) \\ &= \Big(\bigcup_{p \in X} U_p\Big) \cap f^{-1}(W) \end{aligned}$$

$$= \bigcup_{p \in X} \left(U_p \cap f^{-1}(W) \right)$$

$$= \bigcup_{p \in X} f|_{U_p}^{-1}(W).$$

Since each $f|_{U_p}^{-1}(W)$ is open in U_p it is also open in X and consequently $f^{-1}(W)$ is open. $\square$

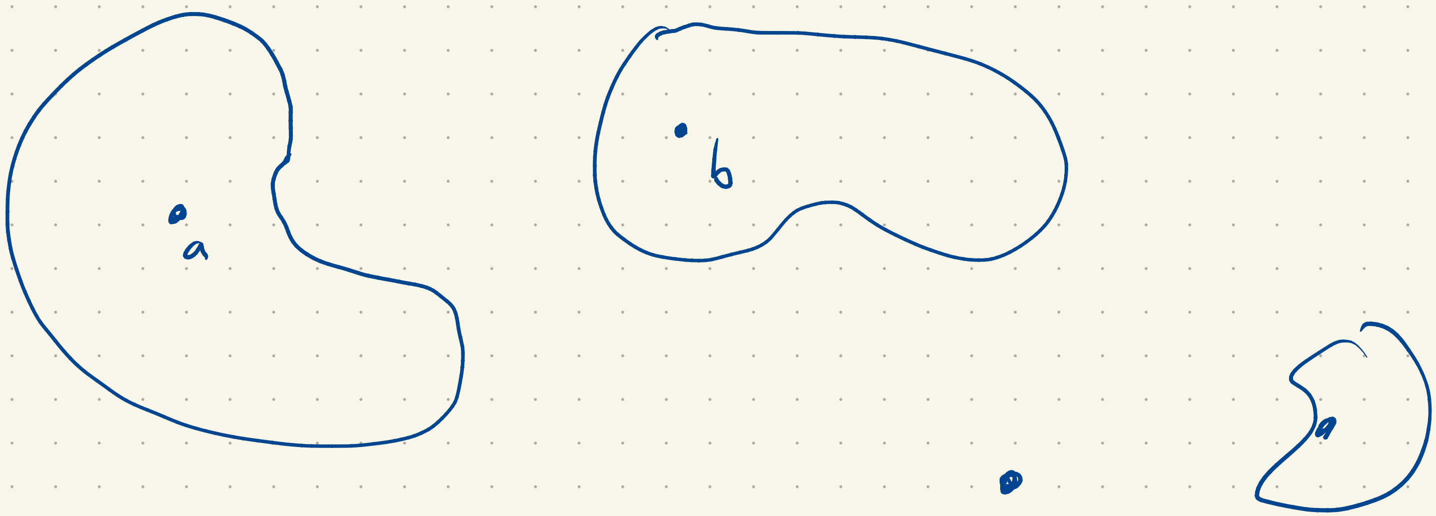
$f|_{U_p} : U_p \to Y$ is <u>continuous</u>

We want topologies that are rich, but not too rich.

Richness will frequently be provided by:

Def: A topological space X is <u>Hausdorff</u> if for all $a, b \in X$ there exist $U_a \in \mathcal{V}(a)$, $U_b \in \mathcal{V}(b)$ such that

$$U_a \cap U_b = \emptyset.$$



Old fashioned notation: Hausdorff $= T_2$

Exercise: Singletons in a Hausdorff space are closed.

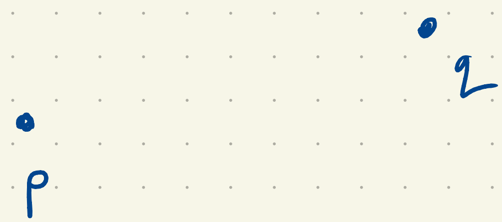
A space is T_1 if singletons are closed.

Cor: In a Hausdorff space finite sets are closed.

Unless X is tiny X and fails Hausdorffness spectacularly.

Metric spaces are Hausdorff

$$r = d(p,q)$$



$$B_{\frac{r}{2}}(p) \cap B_{\frac{r}{2}}(q) = \emptyset$$

Metrizable spaces are Hausdorff.

Def: A sequence $\{x_n\}$ in X converges to x,

$$x_n \to x$$

if for any $U \in \mathcal{U}(x)$ there exists $N \in \mathbb{N}$ such that if $n \geq N$, $x_n \in U$.

Exercise: This is equivalent to the usual definition if X is a metric space.

Prop: In a Hausdorff space limits of sequences are unique.

Bases Recall our description of open sets in metric spaces

a) Balls are our favorite open sets

b) Open sets are unions of balls.

Def: Let (X, τ) be a topological space.

A <u>basis</u> for the topology is a collection $\mathcal{B} \subseteq \tau$ such that for all $U \in \tau$ there exists a subcollection $\mathcal{B}' \subseteq \mathcal{B}$ with

$$U = \bigcup_{B \in \mathcal{B}'} B.$$

Note: to show some collection $\mathcal{B}$ of subsets of X is a basis for the topology you need to:

1) Show that the sets in $\mathcal{B}$ are open

2) Every open set is a union of things in $\mathcal{B}$.

1) is easy to forget.

Exercise: 2) is the same as

"for all $U \in \tau$ and all $p \in U$ there exists $B \in \mathcal{B}$ with $p \in B \subseteq U$"