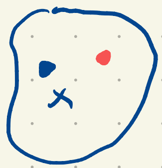
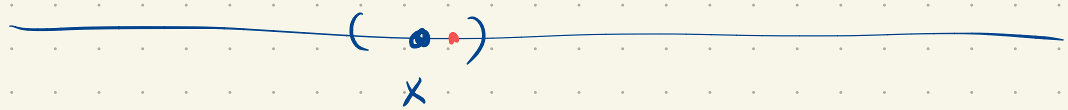
Def: A set A is dense in X if $\bar{A} = X$.

Every point of X is a contact point of A.





Def: Let $x \in X$. A neighborhood of x is an open set containing x. The collection of all such neighborhoods of x is denoted $\mathcal{U}(x)$ and called the **neighborhood base** at x.

Continuity

Metric space version

$$f : X \to Y$$

cts if whenever $x_n \to x$, $f(x_n) \to f(x)$

Alternative:

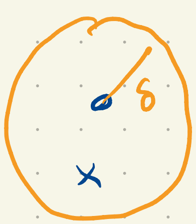
f is cts' if for every $x \in X$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that

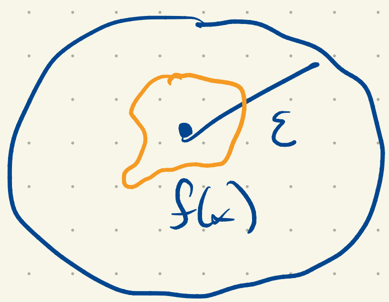
$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

$$B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$$

reminder:

$f^{-1}(W) = \{x \in X : f(x) \in W\}$





Prop: f is cts $\Leftrightarrow$ it is cts'

Pf: Suppose f is cts' and suppose $x_n \to x$ in X.

We need to show $f(x_n) \to f(x)$.

Let $\varepsilon > 0$. Pick $\delta > 0$ so that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

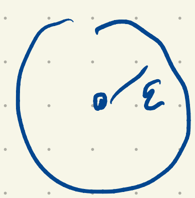
Pick N so that if $n \geq N$ $x_n \in B_\delta(x)$. But then if $n \geq N$, since $x_n \in B_\delta(x)$, $f(x_n) \in B_\varepsilon(f(x))$.

So $f(x_n) \to f(x)$.

Conversely suppose f is not cts'. So there is some $x \in X$ and an $\varepsilon > 0$ such that for all $\delta > 0$ $f(B_\delta(x)) \nsubseteq B_\varepsilon(f(x))$.

But then for each $n \in \mathbb{N}$ we can pick $x_n \in B_{1/n}(x)$ with $f(x_n) \notin B_\varepsilon(f(x))$. But then $x_n \to x$ but

$\underbrace{f(x_n) \not\to f(x)}$.



□

Def: f is cts'' if whenever $U \subseteq Y$ is open, $f^{-1}(U)$ is open in X.

Prop: f is cts' $\iff$ is cts''

Pf: Suppose f is cts'. Let $U \subseteq Y$ be open and pick $x \in f^{-1}(U)$. Since U is open and since $f(x) \in U$, there exists $\varepsilon > 0$ with $B_\varepsilon(f(x)) \subseteq U$. Since f is cts' there exists $\delta > 0$ so that

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \subseteq U.$$

That is, $B_\delta(x) \subseteq f^{-1}(U)$ and $f^{-1}(U)$ is hence open.

Conversely, suppose f is cts''. Let $x \in X$ and pick $\varepsilon > 0$. Let $U = B_\varepsilon(f(x))$ so U is open.

Hence $f^{-1}(U)$ is open and contains x. But then there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(U) = f^{-1}(B_\varepsilon(f(x)))$.

Def: Let X, Y be topological spaces.

We say $f: X \to Y$ is continuous if whenever $U \subseteq Y$ is open, $f^{-1}(U)$ is open in X.

Examples 1) Every continuous function you knew about before taking a topology class.

2) $f: X \to Y$

$f(x) = y_0$ for all x.

(f is a constant function)

$U \subseteq Y$, open $\quad f^{-1}(U) = \begin{cases} \phi & y_0 \notin U \\ X & y_0 \in U \end{cases}$

3) $f: X \to X$

$f(x) = x \quad (f = Id)$

4) A composition of continuous functions is cts.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$$(g \circ f)^{-1}(U) = \{x \in X : g(f(x)) \in U\}$$

$$= \{x \in X : f(x) \in g^{-1}(U)\}$$

open in Z

$$= \{x \in X : x \in f^{-1}(g^{-1}(U))\}$$

$$= f^{-1}(g^{-1}(U))$$

Handy facts

1) $f^{-1}\left(\bigcup_{\alpha\in I} A_\alpha\right) = \bigcup_{\alpha\in I} f^{-1}(A_\alpha)$

2) $f^{-1}\left(\bigcap_{\alpha\in I} A_\alpha\right) = \bigcap_{\alpha\in I} f^{-1}(A_\alpha)$

3) $f^{-1}(A^c) = f^{-1}(A)^c$

1)' $f\left(\bigcup_{\alpha\in I} C_\alpha\right) \stackrel{?}{=} \bigcup_{\alpha\in I} f(C_\alpha)$ yep!

2)' $f\left(\bigcap_{\alpha\in I} C_\alpha\right) \stackrel{?}{=} \bigcap_{\alpha\in I} f(C_\alpha)$  no!

3)' $f(C^c) \stackrel{?}{=} f(C)^c$

$f(x) = y_0$

$f(C^c) = \{y_0\}$

$f(C) = \{y_0\}$,

$f(C)^c = Y \setminus \{y_0\}$

Exercise: Make 2' and 3' correct by changing $=$ to an appropriate inclusion.

Exercise: $f: X \to Y$ is continuous if and only if whenever $V \subseteq Y$ is closed, $f^{-1}(V)$ is closed.

Def: Let τ_1 and τ_2 be two topologies on X.

We say τ_1 is <u>finer</u> than τ_2 (and τ_2 is <u>coarser</u> than τ_1) if $\tau_1 \supseteq \tau_2$.

$$f: X \to Y$$

The finer the topology on X and the courser the topology on Y the easier it is for f to be continuous.

A good topology strikes a ballace between having too few and too many open sets.

$$f: X_{disc} \to Y \quad \text{is always continuous.}$$

$$f: X \to Y_{ind} \quad \text{is always continuous.}$$

$$X_{ind} \xrightarrow{f} \mathbb{R} \xrightarrow{g} X_{disc}$$

Challenge: f and g are continuous iff they are const.

Def: A map $f: X \to Y$ is

<u>open</u> if $f(U)$ is open in Y whenever U is open in X

<u>closed</u> if $f(V)$ is closed in Y whenever V is closed in X.

HW: cts, open, closed are all independent

Def: A map $f: X \to Y$ is a <u>homeomorphism</u> if it is a bijection, is continuous, and has a continuous inverse.